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Cybernetics

An Alternative Transient Solution for Gaver's Parallel System with Repair

Ramaz Khurodze[*], Revaz Kakubava[**], Teimuraz Saghinadze[§], Nino Svanidze[#]

[*]*Academy Member, Georgian Technical University, Tbilisi, Georgia*
[**]*Department of Information Technologies, Georgian Technical University, Tbilisi, Georgia*
[§]*Muskhelishvili Institute of Computational Mathematics, Tbilisi, Georgia*
[#]*Faculty of Exact Science and Education, Batumi Shota Rustaveli State University, Batumi, Georgia*

Gaver's parallel system is investigated using well established method of supplementary variable in conjunction with purely probabilistic reasoning. This results in significant simplifications of reliability analysis of this classical system. Namely, this new approach allows us to forgo solving partial differential equations (Kolmogorov forward equations) of non-classical boundary value problem of mathematical physics with non-local boundary conditions and directly deriving the system's solution in terms of operational calculus. *© 2022 Bull. Georg. Natl. Acad. Sci.*

Semi-Markov process, probabilistic reasoning, Laplace transform, failure, repair

In Mathematical Theory of Reliability and Queuing Theory Semi-Markov processes are extensively used and produce wide range of useful results. Amongst many such processes Gaver's parallel system is of big theoretical and practical importance. Named after D.P. Gaver, JR., the work captured interest of researchers in the field of Reliability Theory [1]. A number of papers, out of which we name only just a few [2,3], taking inspiration from Gaver's parallel system, are testaments to his influence. He also introduced the method of supplementary variable (developed by D.R. Cox for M/G/1 queueing systems [4]) to his research. Ever since then it became one of primary modes on analysis in the field of reliability.

The new method introduced in [5,6] is applied to model describing long-run availability of paralleled systems consisting of two units. Proposed pure probabilistic approach significantly simplifies derivation of Laplace transforms of probability distribution functions, which are of primary interest.

We would like to pay due respect to foundational work by D.P. Gaver, JR., showing transient solution to the system and hopefully making his results more easily accessible to the readers who take interest in semi-Markov processes.

Description of Gaver's Parallel System

In this section we introduce two-unit system as introduced by D.P. Gaver, JR. in his paper [1], commonly called Gaver's Parallel System [2,3]. Only slight alterations are made to original notation to put it in accordance with modern notation.

The system, consisting of two unreliable, repairable units, working in parallel. It is considered to be failed when both units fail. Both units can be in the failed state at the same time and only one of them can be repaired at a time. We introduce the following notations:

$n(t)$ is a random variable representing number of failed units at time t. Life times of the units are random variables with exponential distribution, with rates α_0 and α_1. In particular,

$$\begin{cases} \lim\limits_{h\to 0} \dfrac{\mathbb{P}\{n(t+h)=1 \mid n(t)=0\}}{h} = \alpha_0, \\ \lim\limits_{h\to 0} \dfrac{\mathbb{P}\{n(t+h)=2 \mid n(t)=1\}}{h} = \alpha_1. \end{cases}$$

Repair time σ is a random variable with general cumulative distribution function $G(x)=\mathbb{P}\{x \le \sigma\}$. Hence, the repair rate (hazard function) is $\mu(x)=\dfrac{g(x)}{1-G(x)}$, where $g(x)=G'(x)$. $\xi(t)$ represents the time that has elapsed since the beginning of the current repair job.

$P_0(t)$, $p_1(x,t)$, $p_2(x,t)$ are defined as follows:

$$P_0(t)=\mathbb{P}\{n(t)=0\},$$

$$p_1(x,t)=\lim_{h\to 0}\frac{\mathbb{P}\{n(t)=1, x<\xi(t)\le x+h\}}{h},$$

$$p_2(x,t)=\lim_{h\to 0} h \to 0 \frac{\mathbb{P}\{n(t)=2, x<\xi(t)\le x+h\}}{h}$$

One can derive the following system of equations using elementary probability arguments:

$$\frac{d}{dt}P_0(t)=-\alpha_0 P_0(t)+\int_0^t p_1(x,t)\mu(x)dx, \tag{1}$$

$$\frac{\partial}{\partial t}p_1(x,t)+\frac{\partial}{\partial x}p_1(x,t)=-(\alpha_1+\mu(x))p_1(x,t), \tag{2}$$

$$\frac{\partial}{\partial t}p_2(x,t)+\frac{\partial}{\partial x}p_2(x,t)=\alpha_1 p_1(x,t)-\mu(x)p_2(x,t). \tag{3}$$

Initial and boundary conditions can be obtained in the similar manner:

$$p_1(0,t)=\alpha_0 P_0(t)+\int_0^t p_2(x,t)\mu(x)dx, \tag{4}$$

$$p_2(0,t)=0, \tag{5}$$

$$P_0(0)=1. \tag{6}$$

New Probabilistic Solution

Along the lines of previous works [5,6] we derive transient solution to the system presented in section 2, using purely probabilistic reasoning. Thus, allowing us to forgo solving the system of partial differential consisting of equations (1) and (2).

We will need following definitions of events $A_i(x,t,h)$, $B(x,t)$ and $C(x)$.

$$A_i\left(x,t,h\right)=\left\{n\left(t\right)=i; x<\xi\left(t\right)\le x+h\right\},\ for\ i=1,2,$$

$$B\left(x,t\right)=\left\{n\left(t\right)-n\left(t-x\right)=1; \forall t_0\in\left(t-x,t\right)(0\le n\left(t_0\right)\le 1)\right\},$$

$$C\left(x\right)=\left\{x\le\sigma\right\}.$$

From definition of $A_i(x,t,h)$ it follows that

$$\mathbb{P}\left\{A_i\left(x,t,h\right)\right\}=p_i\left(x,t\right)h+o\left(h\right).$$

Theorem 1. $p_1(x,t)$ can be expressed as:

$$p_1\left(x,t\right)=p_1\left(0,t-x\right)\left(1-G\left(x\right)\right)e^{-\alpha_1 x}. \tag{7}$$

Proof.

$$A_1\left(x,t,h\right)=\left(A_1\left(0,t-x,h\right)\cap B\left(x,t\right)\cap C\left(x\right)\right).$$

All three events on the righthand side are independent, then:

$$\mathbb{P}\left\{A_1\left(x,t,h\right)\right\}=\mathbb{P}\left\{A_1\left(0,t-x,h\right)\cap B\left(x,t\right)\cap C\left(x\right)\right\}.$$

Hence:

$$p_1\left(x,t\right)h+o\left(h\right)=p_1\left(0,t-x\right)h\cdot e^{-\alpha_1 x}\cdot\left(1-G\left(x\right)\right)+o(h).$$

Dividing both sides by h and take the limit $h\to 0$ results in (7).

Theorem 2. $p_2(x,t)$ can be expressed as:

$$p_2\left(x,t\right)=p_1\left(0,t-x\right)\left(1-G\left(x\right)\right)\left(1-e^{-\alpha_1 x}\right). \tag{8}$$

Proof.

$$A_2\left(x,t,h\right)=\left(A_1\left(0,t-x,h\right)\cap\left(B\left(x,t\right)\right)^c\cap C\left(x\right)\right)\cup\left(A_2\left(0,t-x,h\right)\cap C\left(x\right)\right).$$

All three events on the righthand side are independent, then:

$$\mathbb{P}\left\{A_2\left(x,t,h\right)\right\}=\mathbb{P}\left\{\left(A_1\left(0,t-x,h\right)\cap\left(B\left(x,t\right)\right)^c\cap C\left(x\right)\right)\cup\left(A_2\left(0,t-x,h\right)\cap C\left(x\right)\right)\right\}.$$

Hence:

$$p_2\left(x,t\right)h+o\left(h\right)=p_1\left(0,t-x\right)\cdot(1-e^{-\alpha_1 x})\cdot\left(1-G\left(x\right)\right)+p_2\left(0,t-x\right)h\cdot\left(1-G\left(x\right)\right)+o(h).$$

$p_2(0,t-x)$ term vanishes due to boundary condition (5). Lastly, dividing both sides by h and take the limit $h\to 0$ will result in (8).

Using these results, we can find Laplace transform of $P_0(t)$, $p_1(x,t)$ and $p_2(x,t)$ with respect to variable t, denoted with $\overline{P}_0(s)$, $\overline{p}_1(x,s)$ and $\overline{p}_2(x,s)$ respectively.

Expanding $p_1(x,t)$ in (1) using (7) will result in the following expression:

$$\frac{d}{dt}P_0(t) = -\alpha_0 P_0(t) + \int_0^t p_1(0,t-x)\, g(x)\, e^{-\alpha_1 x} dx.$$

Taking Laplace transform of both sides and rearranging terms will result in the following equation and using (6) we get:

$$\overline{P}_0(s) = \frac{1 + \overline{p}_1(0,s)\,\overline{g}(s)}{s + \alpha_0}. \tag{9}$$

Combining (4) and (8) in a similar fashion will result in

$$p_1(0,t) = \alpha_0 P_0(t) + \int_0^t p_1(0,t-x)\left(1 - e^{-\alpha_1 x}\right) g(x)\, dx.$$

Taking again Laplace Transform of both sides and rearranging terms will result in the following expression:

$$\overline{p}_1(0,s) = \frac{\alpha_0 \overline{P}_0(s)}{\overline{g}(s) - 1 - \overline{g}(s + \alpha_1)}. \tag{10}$$

Combining (9) and (10) yields

$$\overline{P}_0(s) = \frac{\overline{g}(s) - 1 - \overline{g}(s + \alpha_1)}{\left(\overline{g}(s) - 1\right)\left(s + \alpha_0\right) - s\overline{g}(s + \alpha_1)}, \tag{11}$$

$$\overline{p}_1(0,s) = \frac{\alpha_0}{\left(\overline{g}(s) - 1\right)\left(s + \alpha_0\right) - s\overline{g}(s + \alpha_1)}. \tag{12}$$

And as a final step obtaining Laplace transform of (7) and (8) combining them one by one with (12) gives the expressions:

$$\overline{p}_1(x,s) = \frac{\alpha_0\left(1 - G(x)\right) e^{-(\alpha_1 + s)x}}{\left(\overline{g}(s) - 1\right)\left(s + \alpha_0\right) - s\overline{g}(s + \alpha_1)}, \tag{13}$$

$$\overline{p}_2(x,s) = \frac{\alpha_0\left(1 - G(x)\right)\left(e^{-sx} - e^{-(\alpha_1 + s)x}\right)}{\left(\overline{g}(s) - 1\right)\left(s + \alpha_0\right) - s\overline{g}(s + \alpha_1)}. \tag{14}$$

Conclusion

Stochastic process described by $n(t)$, on its own, is clearly non-Markovian. But in accordance with method of supplementary variable we redirect our attention towards $(n(t), \xi(t))$. This pair describes two-dimensional Markov process. Writing down Kolmogorov's equation seems like a natural next step. Yet, it turns out they're wholly unnecessary.

As one can see, based on pure probabilistic reasoning and using elementary properties of Laplace transform one can obtain non-trivial results. Our analysis concludes with Laplace transforms of the probabilities describing the system, any practitioner can derive grate interest to the field of Reliability Engineering. As for systems of partial-differential equations (3), (4) we decided to keep them for reasons that are twofold: Firstly, to show the complexities associated with the solution of such system not using our novel method. Secondly, for the reader, if one so desires one can check the consistency of new results with the previous work done by D.P. Gaver, JR.

კიბერნეტიკა

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 რ. ხუროძე*, რ. კაკუბავა**, თ. საღინაძე§, ნ. სვანიძე#

**აკადემიის წევრი, საქართველოს ტექნიკური უნივერსიტეტი, თბილისი, საქართველო*
***საქართველოს ტექნიკური უნივერსიტეტი, ინფორმაციული ტექნოლოგიების დეპარტამენტი, თბილისი, საქართველო*
§ნიკო მუსხელიშვილის სახ. გამოთვლითი მათემატიკის ინსტიტუტი, თბილისი, საქართველო
#ბათუმის შოთა რუსთაველის სახელმწიფო უნივერსიტეტი, ზუსტ მეცნიერებათა და განათლების ფაკულტეტი, ბათუმი, საქართველო

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