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Mathematics

Cauchy Problem with Closed Support of the Data for Quasi-Linear Equation of Mixed Type

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In this paper the Cauchy problem with closed support for quasi-linear hyperbolic equation with admissible parabolic degeneracy is investigated. Solution to the problem is constructed in implicit form. The structure of the domain of definition of the solution is described. We deal with the case, when characteristic curves do not cross the boundary, which is formed due to parabolic degeneration. So, the existence of sub-domain impenetrable for the solution is found. *© 2023 Bull. Georg. Natl. Acad. Sci.*

quasi-linear hyperbolic equation, parabolic degeneracy, Cauchy problem, closed support

Among several problems posed for hyperbolic equations, there are well known problems when we have to find an unknown solution by given values of the solution and its oblique derivative on the support. At the same time, it is also necessary to fulfill the certain conditions regarding initial support. The class of such problems is wide and the known Cauchy problem also belongs to it. Namely, according to the Cauchy problem we have to find the solution of equation by given initial perturbations:

$$u\big|_{t=0} = \tau\ ,\ u_t\big|_{t=0} = \nu\ .$$

After posing such problems it is immediately demanded from the initial support, that every characteristic of the equation must not cross it more than in one point. Also it is demanded from the support, that it must not have the characteristic direction in any point. For the above-mentioned Cauchy problem, the initial support is represented by relation $t = 0$. This initial support has to be subjected to the same requirements. The existence of these requirements is caused by certain reasons. These reasons are connected with common properties of solutions of the equations. If we consider the equations along the characteristic manifolds, we'll see that in major cases these properties themselves emerge from the equations.

According to Hadamard [1], the Cauchy problem may appear correct in some cases of closed initial support. Such problems for linear equations were studied in [2-6].

As it is known, in case of linear equations, the characteristic families are determined by the principal part of equation. The situation is different in case of non-linear equations, when characteristic families cannot be determined in advance, because they depend on an unknown solution u and its derivatives u_x, u_t. This happens in the case of quasi-linear hyperbolic equation

$$au_{xx} + bu_{xt} + cu_{tt} = f . \tag{1}$$

The main coefficients and right-hand part of this equation depend on five x, t, u, u_x, u_t variables. In spite of the fact that in such cases the investigation of the problem gets complicated here at least we have some freedom when posing the problem. We also have the opportunity to modify and generalize the well-known linear problems [7]. Some examples of such non-linear problems are considered in [8, 9]. However, there is no need to generalize and modify the Cauchy problem because it always means to find the solution by given values of it and its non-tangent direction derivative on the support. We emphasize this fact, because in the present paper we are going to speak about the Cauchy problem posed for (1) non-linear equation of special type. As it is well known [10], the hyperbolic class of solutions is determined through its characteristic roots by inequality $\lambda_1 \neq \lambda_2$. The characteristic directions also have to be determined in every point by the following relations: $\frac{dt}{dx} = \lambda_1(x,t,u,u_x,u_t)$, $\frac{dt}{dx} = \lambda_2(x,t,u,u_x,u_t)$. Let us assume that the initial support is given in explicit form by equation $t = f(x)$, where f function is subjected to certain conditions. We speak about these conditions below. In this case we can determine the direction of any characteristic curve which comes out from an arbitrary point of the support.

In order to describe the domain of definition for the solution of the problem we have to write the equations of the characteristic curves of both families. For this purpose, we can use the representations of general integrals [11]. Since the construction of the general integral in closed form is not always possible, here we consider these special type of equations, admitting the construction of general integrals for them. By means of general integral we have to describe both families of characteristics and construct the set of intersection points of different families of characteristic curves. Generally, such sets create the domain inside of which the initial support is situated. Combinations which remain constant in case of the string equation differ from those for non-linear equations. The main difference is that for non-linear equations these combinations depend on an unknown solution and its first derivatives. In the case of aerodynamics these combinations are called Riemannian invariants. Generally, to construct the characteristic invariants and to use them in solving problems for quasi-linear equations is very difficult. We'll try to do it in a case of a concrete equation, which somehow is related with the flow of air and liquid in channels [12]:

$$u_t(u_t - 1)u_{xx} + (u_t - u_x - 2u_x u_t + 1)u_{xt} + u_x(u_x + 1)u_{tt} = 0. \tag{2}$$

This is a hyperbolic equation; however in some cases it admits a parabolic degeneracy. This happens when the characteristic roots of equation (2) are equal, i.e.

$$-\frac{u_x + 1}{u_t} = \frac{u_x}{1 - u_t}.$$

So, the set of hyperbolic solutions is described by relation as follows:

$$u_x - u_t + 1 \neq 0 .$$

Characteristic invariants are described by the systems:

$$\begin{cases} \xi_1 = u + x, \\ \xi_2 = \dfrac{u_x}{u_x - u_t + 1}, \end{cases} \quad \begin{cases} \eta_1 = u - t, \\ \eta_2 = \dfrac{u_x + 1}{u_x - u_t + 1}. \end{cases}$$

By means of characteristic method and Poisson's brackets it is possible to obtain the general integral of the equation (2)

$$f(u+x) + g(u-t) = t, \tag{3}$$

where $f,\ g \in C^2(R)$ are two arbitrary functions.

For equation (2) we studied the Cauchy problem posed on the circle:

$$\gamma: \ \ (x-a)^2 + t^2 = \left(\sqrt{2} - a\right)^2, \ \ a < 0.$$

Using polar coordinates $x = \rho\cos\alpha, \quad t = \rho\sin\alpha$ we can write this problem as follows:

$$\begin{cases} u\big|_{\gamma} = \tau(\alpha), \ \ \alpha \in [0, 2\pi], \\ u_{\rho}\big|_{\gamma} = \nu(\alpha), \ \ t \in [0, 2\pi], \quad \tau, \nu \in C^2[0, 2\pi], \end{cases} \tag{4}$$

where the support of initial data is the following circle:

$$\gamma: \ \ \rho = a\cos\alpha + \sqrt{a^2\cos^2\alpha + 2 - 2\sqrt{2}a}, \quad \alpha \in [0, 2\pi]. \tag{5}$$

Theorem. The Cauchy problem (2), (4) posed on the closed curve (5) is correct.

Proof. If we subject the General integral (3) to initial conditions (4), we obtain the implicit solution for the problem (2), (4):

$$\int_{T_2(u-t)}^{T_1(u+x)} H(\alpha)d\alpha + (2-a)\sin(T_2(u-t)) = t, \tag{6}$$

where

$$H(\alpha) = \frac{(2-a)[((a-1)\sin\alpha - \nu(\alpha))\cos\alpha + \tau'(\alpha)\sin\alpha]}{((a-2)\nu(\alpha) + \tau'(\alpha))\cos\alpha + ((2-a)\nu(\alpha) + \tau'(\alpha))\sin\alpha + a - 2}[\tau'(\alpha) + (a-2)\sin\alpha]$$

and the functions $\alpha = T_1(z)$, $\alpha = T_2(\omega)$ are inverse functions of the following expressions correspondingly:

$$z = \tau(\alpha) + (2-a)\cos\alpha + a, \ \ \omega = \tau(\alpha) + (2-a)\sin\alpha. \tag{7}$$

On the base of (6) it is easy to describe both families of characteristic curves. They are one-parametric families, which come out from arbitrary points of support (5):

$$t = \int_{T_2(c-t-x)}^{c} H(\alpha)d\alpha + (2-a)\sin(T_2(c-t-x)), \tag{8}$$

$$t = \int_{c}^{T_1(c+t+x)} H(\alpha)d\alpha + (2-a)\sin(T_2(c)). \tag{9}$$

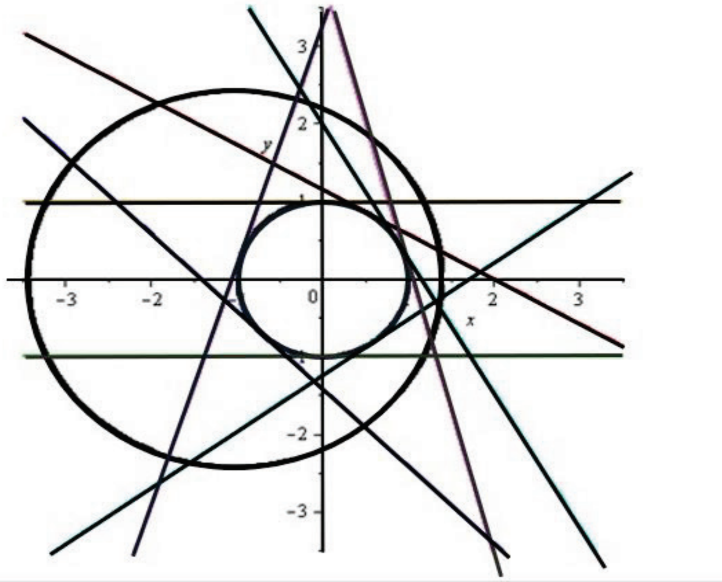

Fig. Closed support of data and envelope for characteristic lines.

As it is known the discriminant curve for the family (8) exists when the following system is solvable in regard to variables x and t:

$$\begin{cases} \int_{T_2(c-t-x)}^{c} H(\alpha)\,d\alpha + (2-a)\sin(T_2(c-t-x)) = t, \\ H(c) - H(T_1(c-t-x))T_1'(c-t-x) + (2-a)\cos(T_2(c-t-x))T_2'(c-t-x) = 0, \end{cases} \tag{10}$$

where the second equation of the system is obtained by derivation of (8) with respect to parameter c.

Analogously, we can write the similar system in the case of family (9):

$$\begin{cases} \int_{c}^{T_1(c+t+x)} H(\alpha)\,d\alpha + (2-a)\sin(T_2(c)) = t, \\ H(T_1(c+t+x))T_1'(c+t+x) - H(c) + (2-a)\cos(T_2(c))T_2'(c) = 0. \end{cases} \tag{11}$$

It is evident that both systems (10), (11) include τ and ν functions and their derivatives. They also include T_1 and T_2 functions and their derivatives. It should be mentioned, that according to (7), T_1 and T_2 are inverse functions of the expressions that depend only on τ .

Among multiple variants of concrete initial functions τ and ν , we have found an example for the families when (8) and (9) discriminant curve is the same circle:

$$x^2 + t^2 = 1. \tag{12}$$

The initial conditions (4) in this case are as follows:

$$u\big|_{\gamma} = u_0 + \left(a\cos\alpha + \sqrt{a^2\cos^2\alpha + 2 - 2\sqrt{2}a}\right)\sin\alpha, \quad \alpha \in [0, 2\pi],$$

$$u_{\rho}\big|_{\gamma} = \sin\alpha, \quad \alpha \in [0, 2\pi],$$

where u_0 is the value of the solution in the arbitrary point of the support, and the characteristic curves are straight lines

$$x\cos\alpha + t\sin\alpha = 1, \quad \alpha \in [0, 2\pi].$$

Each of these straight lines touches the circle (12) at one point (see Figure). This point divides each straight line into two rays. Each of these rays belongs to different families of characteristics. Each line of one family of characteristics crosses the line of other family only one time. At the same time, the circle (12) is the envelope for both families of characteristics. Thus, we have the parabolic degeneracy of the equation on the circle (12), and this parabolic degeneracy forms an area, inside of which the characteristic curves do not enter.

მათემატიკა

კოშის ამოცანა მონაცემთა შეკრული მზიდით შერეული ტიპის კვაზიწრფივი განტოლებისათვის

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