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Informatics

Solving the Linear Ordering Problem Using the Facets (NP=P)

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In the paper presents algorithm of the NP-hard linear ordering problem. For the linear ordering polytope, about 30 main classes of facets (facet-face of maximum dimension) are constructed. From the main facet classes, using three types of mappings, we get about 70 more facet classes. Note that each class of facets contains an exponential number of facets, therefore, when solving the linear ordering problem, a special approach is required, which successfully solved in this work. The linear ordering problem is solved as an integer linear programming problem. Solving the linear programming problem, when obtaining a non-integer solution, we find all the necessary cutting facets using a polynomial algorithm. Next, we add the resulting facets to the linear programming problem and again solve the linear programming problem. This approach to solving the problem continues until now we get an integer solution. Each time, we can find all the necessary facet cuts with the help of the polynomial algorithm. Therefore, we get a polynomial algorithm for solving the linear ordering problem. Consequently NP=P. *© 2023 Bull. Georg. Natl. Acad. Sci.*

NP hard problem, linear ordering problem, polytope, facets, polynomial algorithm

In [1], (m,k) facets are obtained using adjacent integer vertices for a (m,k) non-integer vertex of the polytope B_n, it means that this adjacent integer vertices uniquely determine the facets. In [1], we presented the main facets of the linear ordering polytope P_n. In this paper, from the main facets with the help of three types of mappings, we get the remaining all facets of the linear ordering polytope P_n. Main facets described in [1-10], and at [11,12], mappings were studied that pass the facets of the linear ordering polytope P_n into facets again.

Remark 1. In this paper, we use some results, notation and definitions in [1].

Mappings of the Non-Integer Vertices of the Polytope B_n and the Corresponding Facets of the Polytope P_n

Three types of polytope facet mappings are given, with the help of which we can obtain new facets from known, so that the underlying properties of the facets do not change. We are especially interested in the property: facets are uniquely determined by adjacent integer vertices for a non-integer vertex of the polytope B_n.

Next, we will build facets that are obtained using Mobius ladders of length k, $k \geq 3$ and k odd, where the length of the dicycles in the Mobius ladders is 3 or 4.

Theorem 1. If a non-integer vertex x^0 of the polytope B_n corresponding to a facet obtained with a Mobius ladders of length k, $k \geq 3$ and k odd, where the length of the dicycles in the Mobius ladders is 3 or 4, then this non-integer vertex can be recognized using a polynomial algorithm.

Theorem 2. If a non-integer vertex of the polytope B_n corresponding to a facet, which is obtained with the help of Mobius ladders of length k, $k \geq 3$ and k odd, where the length of the dicycles in the Mobius ladders is equal to 3 or 4, then all exact facet cuts are constructed using the polynomial algorithm.

We present now a new facet generating method – the rotation method [11, 12] which consists in the following.

Let P_n be a linear ordering polytope in R^{n^2-n}. Introduce an affine mapping φ of R^{n^2-n} into itself. φ we call the rotation mapping of P_n, $vertP_n = vert\varphi(P_n)$. Rotation mapping realize one-to-one mapping of the facets of P_n and corresponding non integer vertices of B_n into the facets of P_n and corresponding non- integer vertices of B_n. Hence, having the facet inequality $ax \leq a_0$ of P_n into the facet $a\varphi(x) \leq a_0$ of P_n.

It appears that the mapping $\varphi_r : R^{n^2-n} \to R^{n^2-n}$ is productive for new facets generation from known ones for any $r = 1,\ldots,n.$ This mapping is suggested and is defined like this

$$\varphi_r : x_{ir} = x'_{ri},\ x_{ri} = x'_{ir},\ i \neq r,\ i = 1,...,n;$$

$$x_{ij} = x'_{ij} + x'_{jr} - x'_{ir},\ i \neq j,\ i \neq r,\ j \neq r,\ i,j = 1,...,n.$$

Theorem 3. If the inequality $\sum\sum a_{ij}x_{ij} \leq a_0,\ r \in \{1,\ldots,n\}$ defines a facet for P_n, then the inequality

$$\sum_{i=1,i\neq j,i\neq r}^{n}\ \sum_{j=1,j\neq r}^{n} a_{ij}(x_{ij} + x_{jr} - x_{ir}) + \sum_{i=1,i\neq r}^{n}(a_{ir}x_{ri} + a_{ri}x_{ir}) \leq a_0,$$

defines a facet for P_n.

Definition 1. If a non-integer vertex x^0 of the polytope B_n has at least one exact facet cut, then a non-integer vertex x^0 is called faceted non-integer vertex of the polytope B_n.

Theorem 4. If x^0 a faceted non-integer vertex of the polytope B_n but is not the (m,k) non-integer vertex and when mapping φ_r, $r \in \{1,\ldots,n\}$, we obtain a (m,k) non-integer vertex x^1, then for x^1 constructed exact facet cuts after the mapping φ_r^{-1} pass to the exact facet cuts for the faceted non-integer vertex x^0.

Example 1. x^0 a faceted integer vertex of a polytope is shown in Fig. 1. Aftzer the mapping φ_3, we obtain (m,k) non-integer vertex x^1 shown in Fig. 2. We know the exact facet cuts for the (m,k) non-integer vertex shown in Fig. 3 (see [1]). The facet shown in Fig. 3 after the mapping φ_3^{-1}, go over the facet shown in Fig. 4, which is the exact facet cuts for the x^0 facet non-integer vertex of the polytope B_6.

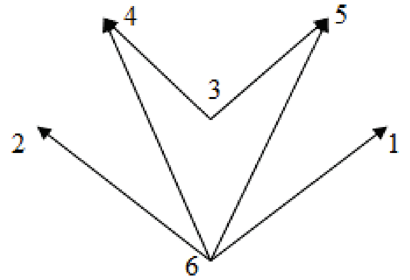

Fig. 1. Non-integer vertex x^0 of B_n.

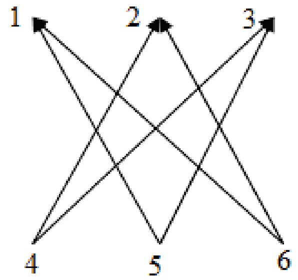

Fig. 2. Non-integer vertex x^1 of B_n.

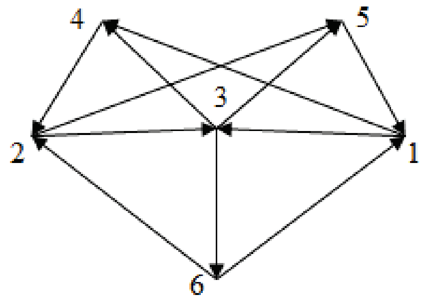

Fig. 4. Facet of P_n obtained after the mapping φ_3^{-1}.

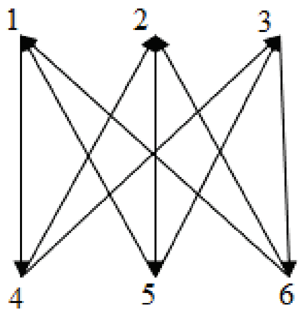

Fig. 3. Facet of P_n.

Similarly mapping φ_r are used mapping ψ_{n+1}. Mapping $\psi_{n+1}: R^{n^2-n} \to R^{(n+1)^2-n-1}$, having the following form:

$$\psi_{n+1}: x_{ij} = x'_{ij} + x'_{jn+1} - x'_{in+1}, \quad i \neq j, \quad i,j = 1,\ldots,n$$

also used when obtaining new facets of the linear ordering polytope P_{n+1} and the corresponding new non-integer vertices of the polytope B_{n+1}.

Theorem 5. If the inequality $\sum\sum a_{ij}x_{ij} \le a_0,\ r \in \{1,\ldots,n\}$ defines a facet for P_n, then the inequality

$$\sum_{i=1, i\neq j}^{n} \sum_{j=1}^{n} a_{ij}(x_{ij} + x_{jn+1} - x_{in+1}) \le a_0 \qquad (1)$$

defines a facet for P_{n+1}.

Theorem 6. If a non-integer vertex x^0 of the polytope B_{n+1}, for which the exact facet cut has the form (1), then this non-integer vertex can be recognized using a polynomial algorithm.

Theorem 7. If a non-integer vertex x^0 of the polytope B_{n+1}, for which the exact facet cut has the form (1) and using mapping $\psi_{n+1}^{-1}\left(x^0\right)$, the x^0 non-integer vertex to go to a non-integer vertex x^1, then for x^1 constructed exact facet cuts, after the mapping ψ_{n+1}, they pass to the exact facet cuts for the non-integer vertex x^0.

As already noted in [1], the (m,k) facets are the main facets of the polytope P_n.

Theorem 8. If we have all the (m,k) facets of the polytope P_n, then with the help of the generalization and the above mapping, we can get all the other facets of the polytope P_n.

Theorem 9. If we have all (m,k) non-integer vertices of the polytope B_n, then after the above mapping, we can get all the other faceted non-integer vertices of the polytope B_n.

Non-Integer Vertices of the Polytope B_n, which are not Faceted Non-Integer Vertices

Let a x^0 non-integer vertex of a polytope B_n on the set of indices Z_n and let $V_1, V_2, ..., V_h \subset Z_n$, pair wise disjoint subsets, $V_1 \cup V_2 \cup ... \cup V_h = Z_n$, then any non-integer vertex of the polytope B_n can be represent as follows: $D_{V_1} D_{V_2} ... D_{V_h}$, where D_{V_s}, $s = 1, 2, ..., h$ is the linear order of an integer vertex or the graph corresponding to a non-integer vertex on the set V_s, $s = 1, 2, ..., h.$

If D_{V_s}, $s = 1, 2, ..., h$ is a graph corresponding to a non-integer vertex x_{0s} on the set V_s, then we will study the non-integer vertex x_{0s} of the polytope B_n separately.

Definition 2. If x^0 non-integer vertex of the polytope B_n corresponds to more than one graph D_{V_s}, $s = 1, 2, ..., h$ of non-integer vertices, then x^0 we call a multiple non-integer vertex of the polytope B_n.

Definition 3. If x^0 non-integer vertex of the polytope B_n corresponds to only one graph of a non-integer vertex, then x^0 we will call a single non-integer vertex of the polytope B_n.

Next, consider a non-integer vertex of a special type of polytope B_n.

We introduce the notation $x_{ij} + x_{jk} - x_{ik} = (i, j, k),\ i \neq j,\ i \neq k,\ j \neq k,\ i, j, k = 1, ..., n.$

Example 2. We consider non-integer vertex x^0 of the polytope B_n, at the set $\{1,\ ...,\ 8\}$, where denominators of the non-integer coordinates equal to 3.

$$x^0 = (x_{12} = 0, x_{13} = 0, x_{24} = 1, x_{26} = 1, x_{34} = 1, x_{37} = 1, x_{46} = 0, x_{48} = 0, x_{75} = 0, x_{78} = 0, x_{68} = 0,$$
$$x_{58} = 0, x_{17} = \frac{2}{3}, x_{15} = \frac{1}{3}, x_{18} = \frac{1}{3}, x_{16} = \frac{1}{3}, x_{28} = \frac{1}{3}, x_{14} = \frac{1}{3}, x_{27} = \frac{2}{3}, x_{36} = \frac{1}{3}, x_{25} = \frac{1}{3},$$
$$x_{23} = \frac{2}{3}, x_{38} = \frac{1}{3}, x_{35} = \frac{2}{3}, x_{45} = \frac{1}{3}, x_{76} = \frac{1}{3}, x_{65} = \frac{1}{3}).$$

The point x^0 satisfies the following system of equalities:

$x_{12} = 0, x_{13} = 0, x_{43} = 0, x_{73} = 0;$

$(1,\ 2,\ 7) = 0,\ (1,\ 4,\ 7) = 0,\ (4,\ 7,\ 5) = 0;$

$(2,\ 7,\ 3) = 0,\ (2,\ 3,\ 6) = 0 \Leftrightarrow (2,\ 7,\ 6) = 0,\ (7,\ 3,\ 6) = 0;$

$(2,\ 5,\ 3) = 0,\ (2,\ 3,\ 6) = 0 \Leftrightarrow (2,\ 5,\ 6) = 0,\ (5,\ 3,\ 6) = 0;$

$(2,\ 5,\ 4) = 0,\ (2,\ 4,\ 6) = 0 \Leftrightarrow (2,\ 5,\ 6) = 0,\ (5,\ 4,\ 6) = 0;$

$(1,\ 3,\ 6) = 0,\ (1,\ 6,\ 8) = 0 \Leftrightarrow (1,\ 3,\ 8) = 0,\ (3,\ 6,\ 8) = 0;$

$(1,\ 4,\ 6) = 0,\ (1,\ 6,\ 8) = 0 \Leftrightarrow (1,\ 4,\ 8) = 0,\ (4,\ 6,\ 8) = 0;$

$(2,\ 5,\ 8) = 0,\ (1,\ 2,\ 8) = 0 \Leftrightarrow (1,\ 2,\ 5) = 0,\ (1,\ 5,\ 8) = 0;$

$x_{46}=0, x_{62}=0 \Leftrightarrow x_{42}=0,(4,6,2)=0;$

$x_{46}=0, x_{68}=0 \Leftrightarrow x_{48}=0,(4,6,8)=0;$

$x_{75}=0, x_{58}=0 \Leftrightarrow x_{78}=0,(7,5,8)=0.$

x^0 non-integer vertex is adjacent to the next integer vertex x^1 of the polytope B_n:

$$x^0=(x_{12}=0, x_{13}=0, x_{24}=1, x_{26}=1, x_{34}=1, x_{37}=1, x_{46}=0, x_{48}=0, x_{75}=0, x_{78}=0, x_{68}=0,$$
$$x_{58}=0, x_{17}=\frac{1}{2}, x_{15}=\frac{1}{2}, x_{18}=\frac{1}{2}, x_{16}=\frac{1}{2}, x_{28}=\frac{1}{2}, x_{14}=\frac{1}{2}, x_{27}=\frac{1}{2}, x_{36}=\frac{1}{2}, x_{25}=\frac{1}{2},$$
$$x_{23}=\frac{1}{2}, x_{38}=\frac{1}{2}, x_{35}=\frac{1}{2}, x_{45}=\frac{1}{2}, x_{76}=\frac{1}{2}, x_{65}=\frac{1}{2}).$$

The point x^1 satisfies the following system of equalities:

$x_{12}=0, x_{13}=0, x_{43}=0, x_{73}=0, x_{53}=0;$

$(1,\ 2,\ 7)=0,\ (1,\ 4,\ 7)=0,\ (4,\ 7,\ 5)=0;$

$(2,\ 7,\ 3)=0,\ (2,\ 3,\ 6)=0 \Leftrightarrow (2,\ 7,\ 6)=0,\ (7,\ 3,\ 6)=0;$

$(2,\ 5,\ 3)=0,\ (2,\ 3,\ 6)=0 \Leftrightarrow (2,\ 5,\ 6)=0,\ (5,\ 3,\ 6)=0;$

$(2,\ 5,\ 4)=0,\ (2,\ 4,\ 6)=0 \Leftrightarrow (2,\ 5,\ 6)=0,\ (5,\ 4,\ 6)=0;$

$(1,\ 3,\ 6)=0,\ (1,\ 6,\ 8)=0 \Leftrightarrow (1,\ 3,\ 8)=0,\ (3,\ 6,\ 8)=0;$

$(1,\ 4,\ 6)=0,\ (1,\ 6,\ 8)=0 \Leftrightarrow (1,\ 4,\ 8)=0,\ (4,\ 6,\ 8)=0;$

$(2,\ 5,\ 8)=0,\ (1,\ 2,\ 8)=0 \Leftrightarrow (1,\ 2,\ 5)=0,\ (1,\ 5,\ 8)=0;$

$x_{46}=0, x_{62}=0 \Leftrightarrow x_{42}=0,(4,6,2)=0;$

$x_{46}=0, x_{68}=0 \Leftrightarrow x_{48}=0,(4,6,8)=0;$

$x_{75}=0, x_{58}=0 \Leftrightarrow x_{78}=0,(7,5,8)=0.$

x^1 non-integer vertex of the polytope B_n is not a faceted non-integer vertex, however, the last system of equalities contains the following subsystem of equalities:

$x_{12}=0, x_{13}=0, x_{62}=0, x_{68}=0, x_{58}=0, x_{53}=0;$

$(5,\ 3,\ 6)=0,\ (5,\ 6,\ 2)=0 \Leftrightarrow (5,\ 3,\ 2)=0,\ (3,\ 6,\ 2)=0;$

$(1,\ 3,\ 6)=0,\ (1,\ 6,\ 8)=0 \Leftrightarrow (1,\ 3,\ 8)=0,\ (3,\ 6,\ 8)=0;$

$(1,\ 2,\ 5)=0,\ (1,\ 5,\ 8)=0 \Leftrightarrow (1,\ 2,\ 8)=0, (2,\ 5,\ 8)=0;$

on the set of indices $\{1,\ 6,\ 5,\ 8,\ 3,\ 2\}$, whose solution is a faceted integer vertex x_{01} of the polytpe B_n. For an non-integer vertex x_{01} we can write out the exact facet cut:

$$x_{18}+x_{63}+x_{52}-x_{13}-x_{62}-x_{58}-x_{12}-x_{68}-x_{53} \le 1.$$

In this case, through adjacent non-integer vertices, we can go to the faceted non-integer vertex x_{01}. By cutting off x_{01} with exact facet cutting, we also cut off x^0. For x^0 we can write out more 11 exact faceted cuts. Then, using a polynomial algorithm, through adjacent non-integer vertices, we can pass to non-integer vertices containing faceted non-integer vertices.

Theorem 10. Let x^0 non-integer vertex of a polytope B_n on the set of indices $\{1,2,\dots,n\}$, where the denominator of non-integer coordinates is equal to s. Cutting off the obtained faceted non-integer vertices (their number is $(s-1)n$), using exact facet cuttings, we also cut off the non-integer vertex x^0.

There are also other special non-integer vertices of the polytope B_n that have been successfully studied, but they have not yet been published.

Algorithm for Solving the Linear Ordering Problem

Let x_{uv} non-integer vertex of the polytope B_n, on the set of indices N_{uv}, $u=1,2,\dots,d$, $v=1,...,b_{uv}$, where u is the numbering of cycles when solving the problem, v is the numbering of the added facets at values u; M_{uv} is the set of facets of the polytope B_n, corresponding to the non-integer vertex x_{uv}; M_{uv}^0 is the set of facets of the polytope B_n corresponding to a non-integer vertex x_{uv}^0 of the polytope B_n.

Step 1. Solving the linear programming problem

$$\sum_{i=1,i\neq j}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \to \max;$$

$$0 \le x_{ij} \le 1,\ x_{ij}+x_{ji}=1,\ i\neq j,\ i,j=1,...,n;$$

$$0 \le x_{ij}+x_{jk}-x_{ik} \le 1,\ i\neq j,\ i\neq k,\ j\neq k,\ i,j,k=1,...,n.$$

If we get an integer vertex, then go to step 8. If we get a non-integer vertex x^0, then go to the next step.

Step 2. If x^0 multiple non-integer vertex of the polytope B_n, then we select single non-integer vertices.

Step 3. From single non-integer vertices of the polytope B_n, select faceted non-integer vertices.

Step 4. x_{uv}^{01}, $v=1,...,b$ faceted non-integer vertices.

If x_{uv}^{01}, $v\in\{1,...,b\}$ faceted non-integer vertex of type (7) from Theorem 9, then we construct the corresponding exact facet cuts and add them to the system of equalities (2) and go to step 6.

Step 5. If x_{uv}^{01}, $v\in\{1,...,b\}$ faceted integer vertex is not of type (7), x_{uv}^{01}, $v\in\{1,...,b\}$ corresponds to a facet, which is obtained using the Mobius ladders of length k, $k\ge 3$ and k odd, where the length of the dicycles in the Mobius ladders is equal to 3 or 4, then using Theorem 2 we construct all exact facet cuts and add them to the system of equalities (2) and go to step 6.

If x_{uv}^{01}, $v\in\{1,...,b\}$ is a faceted non-integer vertex from Theorem 7, then using the mapping ψ_r, we construct all exact facet cuts and add them to the system of equalities (2) and go to step 6.

For x_{uv}^{01}, $v\in\{1,...,b\}$, using the mapping φ_r and Theorem 4, we build all exact facet cuts and add them to the system of equalities (2) and go to step 6.

Step 6. Solving the Linear Programming Problem

$$\sum_{i=1,i\neq j}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \to \max$$

$$\begin{aligned}&0 \le x_{ij} \le 1,\ x_{ij}+x_{ji}=1,\ i\neq j,\ i,j=1,...,n;\\&0 \le x_{ij}+x_{jk}-x_{ik} \le 1,\ i\neq j,\ i\neq k,\ j\neq k,\ i,j,k=1,...,n;\\&f_{uv}(x)\le a_{uv},\ \ u=1,2,\dots,d_u,\ \ v=1,2,\dots,b_{uv}.\end{aligned} \tag{2}$$

If we obtain an integer vertex, then go to step 8.

Step 7. If we obtain a non-integer vertex x_{uv} and if $N_{uv-1} \subset N_{uv}$, then $M_{uv}^0 = M_{uv} \cup M_{uv-1}^0$, with the help M_{uv}^0 we determine x_{uv}^0 and go to Step 2;

If $N_{uv-1} \supset N_{uv}$, then on the set of indices N_{uv} we define the set of facets M_{uv}^0, calculate the non-integer vertex x_{uv}^0, and go to step 2.

Step 8. We get the optimal solution.

Next, a simple example of constructing exact facet cuts for the solved problem is given.

Example 3. Solving a linear programming problem

$$\sum_{i=1, i\neq j}^{6} \sum_{j=1}^{6} c_{ij} x_{ij} \to \max;$$

$$0 \le x_{ij} \le 1,\ \mathrm{n}_{ij} + x_{ji} = 1,\ i \neq j,\ i, j = 1, \ldots, n;$$

$$0 \le x_{ij} + x_{jk} - x_{ik} \le 1,\ i \neq j,\ i \neq k,\ j \neq k,\ i, j, k = 1, \ldots, n.$$

We get a non-integer vertex

$$x_{11}^0 = (x_{i_1i_2} = \frac{1}{2}, x_{i_1i_3} = \frac{1}{2}, x_{i_2i_3} = \frac{1}{2}, x_{j_1j_2} = \frac{1}{2}, x_{j_1j_3} = \frac{1}{2}, x_{j_2j_3} = \frac{1}{2},\ \ x_{i_1j_1} = \frac{1}{2}, x_{i_2j_2} = \frac{1}{2}, x_{i_3j_3} = \frac{1}{2},$$
$$x_{i_sj_d} = 0, s \neq d, s, d = 1, 2, 3)$$

of polytope B_n.

x_{11}^0 satisfies the system of linear equalities

$$x_{i_1j_2} = 0, x_{i_2j_3} = 0, x_{i_3j_1} = 0, x_{i_1j_3} = 0, x_{i_2j_1} = 0, x_{i_3j_2} = 0,$$
$$(i_1, j_2, i_2) = 0, (i_1, i_2, j_1) = 0 \Leftrightarrow (i_1, j_2, j_1) = 0\ , (j_2, i_2, j_1) = 0;$$
$$(i_1, j_3, i_3) = 0, (i_1, i_3, j_1) = 0 \Leftrightarrow (i_1, j_3, j_1) = 0\ , (j_3, i_3, j_1) = 0;$$
$$(i_2, j_3, i_3) = 0, (i_2, i_3, j_2) = 0 \Leftrightarrow (i_2, j_3, j_2) = 0\ , (j_3, i_3, j_2) = 0.$$

Further, to the linear programming problem, we add the exact facet cuts for a non-integer vertex x^0 :

$$x_{i_1j_1} + x_{i_2j_2} + x_{i_3j_3} - x_{i_1j_2} - x_{i_2j_3} - x_{i_3j_1} - x_{i_1j_3} - x_{i_2j_1} - x_{i_3j_2} \le 1$$

and solve the linear programming problem again.

Suppose the objective function is such that we obtain the solve:

$$x_{11} = (x_{i_1i_2} = \frac{2}{3}, x_{i_1i_3} = \frac{2}{3}, x_{i_1i_4} = \frac{1}{3}, x_{i_2i_3} = \frac{2}{3}, x_{i_2i_4} = \frac{1}{3}, x_{i_3i_4} = \frac{1}{3}, x_{j_1j_2} = \frac{2}{3}, x_{j_1j_3} = \frac{2}{3},$$
$$x_{j_1j_4} = \frac{1}{3}, x_{j_2j_3} = \frac{2}{3}, x_{j_2j_4} = \frac{1}{3}, x_{j_3j_4} = \frac{1}{3}, x_{i_1j_1} = \frac{1}{3}, x_{i_2j_2} = \frac{1}{3}, x_{i_3j_3} = \frac{1}{3}, x_{i_4j_4} = \frac{2}{3},$$
$$x_{i_sj_d} = 0,\ s \neq d,\ s, d = 1, 2, 3.4).$$

Satisfying the following system of equalities:

$x_{i_1j_2} = 0,\ x_{i_2j_3} = 0,\ x_{i_3j_4} = 0,\ x_{i_4j_1} = 0,\ x_{i_1j_3} = 0,\ x_{i_2j_4} = 0,\ x_{i_3j_1} = 0,\ x_{i_4j_2} = 0,$

$x_{i_1j_4} = 0, x_{i_2j_1} = 0, x_{i_3j_2} = 0, x_{i_4j_3} = 0;$

$(i_1, j_2, i_2) = 0, (i_1, j_2, j_1) = 0;$

$(i_1, j_3, i_3) = 0, (i_1, j_3, j_1) = 0;$

$(i_2, j_3, i_3) = 0, (i_2, j_3, j_2) = 0;$

$(i_1, j_4, i_4) = 0, (i_1, i_4, j_1) = 0 \Leftrightarrow (i_1, j_4, j_1) = 0, (j_4, i_4, j_1) = 0;$

$(i_2, j_4, i_4) = 0, (i_2, i_4, j_2) = 0 \Leftrightarrow (i_2, j_4, j_2) = 0, (j_4, i_4, j_2) = 0;$

$(i_3, j_4, i_4) = 0, (i_3, i_4, j_3) = 0 \Leftrightarrow (i_3, j_4, j_3) = 0, (j_4, i_4, j_3) = 0;$

$x_{i_1 j_1} + x_{i_2 j_2} + x_{i_3 j_3} - x_{i_1 j_2} - x_{i_2 j_3} - x_{i_3 j_1} - x_{i_1 j_3} - x_{i_2 j_1} - x_{i_3 j_2} = 1.$

Next we find $M_{21}^0 = M_{11} \cup M_{11}^0$, having the following form:

$x_{i_1 j_2} = 0, x_{i_2 j_3} = 0, x_{i_3 j_4} = 0, x_{i_4 j_1} = 0, x_{i_1 j_3} = 0, x_{i_2 j_4} = 0, x_{i_3 j_1} = 0, x_{i_4 j_2} = 0,$

$x_{i_1 j_4} = 0, x_{i_2 j_1} = 0, x_{i_3 j_2} = 0, x_{i_4 j_3} = 0;$

$(i_1, j_2, i_2) = 0, (i_1, i_2, j_1) = 0 \Leftrightarrow (i_1, j_2, j_1) = 0 , (j_2, i_2, j_1) = 0;$

$(i_1, j_3, i_3) = 0, (i_1, i_3, j_1) = 0 \Leftrightarrow (i_1, j_3, j_1) = 0 , (j_3, i_3, j_1) = 0;$

$(i_1, j_4, i_4) = 0, (i_1, i_4, j_1) = 0 \Leftrightarrow (i_1, j_4, j_1) = 0, (j_4, i_4, j_1) = 0;$

$(i_2, j_3, i_3) = 0, (i_2, i_3, j_2) = 0 \Leftrightarrow (i_2, j_3, j_2) = 0 , (j_3, i_3, j_2) = 0;$

$(i_2, j_4, i_4) = 0, (i_2, i_4, j_2) = 0 \Leftrightarrow (i_2, j_4, j_2) = 0, (j_4, i_4, j_2) = 0;$

$(i_3, j_4, i_4) = 0, (i_3, i_4, j_3) = 0 \Leftrightarrow (i_3, j_4, j_3) = 0, (j_4, i_4, j_3) = 0.$

Solving the last system of equalities, we find a non-integer vertex x_{21}^0 of the polytope B_n:

$$x_{21}^0 = (x_{i_1 i_2} = \frac{1}{2}, x_{i_1 i_3} = \frac{1}{2}, x_{i_1 i_4} = \frac{1}{2}, x_{i_2 i_3} = \frac{1}{2}, x_{i_2 i_4} = \frac{1}{2}, x_{i_3 i_4} = \frac{1}{2}, x_{j_1 j_2} = \frac{1}{2}, x_{j_1 j_3} = \frac{1}{2},$$
$$x_{j_1 j_4} = \frac{1}{2}, x_{j_2 j_3} = \frac{1}{2}, x_{j_2 j_4} = \frac{1}{2}, x_{j_3 j_4} = \frac{1}{2}, x_{i_1 j_1} = \frac{1}{2}, x_{i_2 j_2} = \frac{1}{2}, x_{i_3 j_3} = \frac{1}{2}, x_{i_4 j_4} = \frac{1}{2},$$
$$x_{i_s j_d} = 0,\ s \neq d,\ s, d = 1, 2, 3.4).$$

The non-integer vertex x_{21}^0 of the polytope B_n is a non-integer vertex of type (7) from Theorem 9, which means that for x_{21}^0 we know the corresponding exact facet cuts that have the form:

$$x_{i_1 j_1} + x_{i_2 j_2} + x_{i_3 j_3} + x_{i_4 j_4} - x_{i_1 j_2} - x_{i_2 j_3} - x_{i_3 j_4} - x_{i_4 j_1} - x_{i_1 j_3} - x_{i_2 j_4} - x_{i_3 j_1} - x_{i_4 j_2}$$
$$- x_{i_1 j_4} - x_{i_2 j_1} - x_{i_3 j_2} - x_{i_4 j_3} \leq 1,$$
$$2(x_{i_1 j_1} + x_{i_2 j_2} + x_{i_3 j_3} + x_{i_4 j_4}) - x_{i_1 j_2} - x_{i_2 j_3} - x_{i_3 j_4} - x_{i_4 j_1} - x_{i_1 j_3} - x_{i_2 j_4} - x_{i_3 j_1} - x_{i_4 j_2}$$
$$- x_{i_1 j_4} - x_{i_2 j_1} - x_{i_3 j_2} - x_{i_4 j_3} \leq 3.$$

We add these inequalities to the constraints of the linear programming problem and solve the linear programming problem again.

Note that only part of the main facet classes has been published. The unpublished main facet classes are constructed similarly to already published facets. In this paper, when solving the linear ordering problem, the transition to a non-integer vertex of the polytope B_n is done successfully, if we obtain faceted non-integer vertex, directly or using the above mappings, we find exact facets, if we obtain special classes of non-integer vertex of a polytope B_n, then with the help of a polynomial algorithm, exact facets are constructed. Thus, Theorem 11 is true.

Theorem 11. Above the presented, the algorithm for solving the linear ordering problem is polynomial. Hence we obtain NP=P.

Remark 2. The main part of these results, about 30 papers, was received long ago, but has not yet been published, because I have no opportunity to work. I mainly do with other things. If this topic will finance commercially or otherwise, then I will publish the articles.

ინფორმატიკა

წრფივი გადაადგილებების ამოცანის ამოხსნა ფასეტების საშუალებით (NP=P)

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განვიხილავთ წრფივი გადაადგილებების ამოცანას, როგორც წრფივი მთელრიცხვა პროგრამირების ამოცანას. წრფივი პროგრამირების ამოცანის ამოხსნისას არამთელი ამონახსნის მიღების შემთხვევაში ვპოულობთ ყველა აუცილებელ ფასეტურ კვეთებს პოლინომიური ალგორითმის გამოყენებით. შემდეგ, ჩვენ ვამატებთ მიღებულ ფასეტურ უტოლობებს წრფივი პროგრამირების ამოცანას და კვლავ ვხსნით. ამოცანის ამოხსნის ასეთი მიდგომა გრძელდება მანამ, სანამ არ მივიღებთ მთელ ამოხსნას. ყოველ ჯერზე, ჩვენ ვპოულობთ ყველა აუცილებელ ფასეტურ კვეთებს პოლინომიური ალგორითმის გამოყენებით. მაშასადამე, მივიღებთ პოლინომიურ ალგორითმს წრფივი გადაადგილებების ამოცანისთვის.

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