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Mathematics

The Consistent Estimators in Banach Space of Measures for Haar Statistical Structures

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Statistical methods should be used to determine probabilistic characteristics. Among the problems of statistics there is a class of problems, in which the number of observations is unique. Despite the uniqueness of observations in many cases it is possible to determine reliably the values of unknown distribution parameters or to reliably choose one of infinite numbers of competing hypotheses about the exact form of the distribution. In the case when a parameter is reliably determined by one observation, it is said that for it there exists a consistent estimate of parameter. In the paper, we define Haar statistical structure. Necessary and sufficient conditions for existence of consistent estimators of parameters in Banach space of measures of the statistical Haar structure are proved.
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consistent estimators, singular, weakly separable, strongly separable, Banach space of measures

Let (E,S) be a measurable space with a given family of probability measures $\{\mu_i, i \in I\}$, where I be the set of parameter. Some definitions from ([1-5]).

By (ZFC) we denote the formal system of Zeremevo Francal with the addition of axiom of choice (AC) i.e. $(ZFC)=(ZF)\S(AC)$ By $(ZFC)\S(CH)$ we denote the theory with the addition of continuum hypothesis $(CH): 2^{\chi_0}=\omega_1$, where ω_1 denotes the first uncountable cardinal number, and by $(ZFC)\S(MA)$ we denote the theory with the addition of Martin's axiom (MA). It is known that in the theory $(ZFC)\S(CH)$ Martin's axiom (MA) is automatically satisfied. It is well known that Martin's axiom (MA) is much weaker than the continuum hypothesis (CH). Moreover, the negation of the continuum hypothesis (TCH) is compatible with Martin's axiom (see [1]).

An object $\{E, S, \mu_i, i \in I\}$ is called a statistical structure. It is known that, if a statistical structure $\{E, S, \mu_i, i \in I\}$ admit weakly or strongly consistent estimators of parameters, then this statistical structure is orthogonal (see [2]).

The notions and corresponding construction of weak separability and strong separability were introduced and studied by A. Skorokhod (see [3]), Z. Zerakidze constructs of strongly separable statistical structure that this statistical structure does not admin a consistent estimators of parameters in the theory (ZF) (see [4]). A. Skorokhod proved that in theory $(ZFC)\S(CH)$ an arbitrary weakly separable statistical structure, whose cardinality of the continuum, is strongly separable (see [4]). Z. Zerakidze proved in the theory $(ZFC)\S(MA)$ Borell weakly separable statistical structure, whose cardinality is not greater than the cardinality of the continuum is strongly separable (see [5]).

This article is devoted to be the question of the existence of consistent estimators of parameters for Haar statistical structures.

Consistent Estimators of Parameters

Let (E, S) be a measurable space, the following definitions are taken from the works [1-5].

Definition 1. Let E be an arbitrary locally compact and σ − compact topological group and $B(E)$ is a σ- algebra of subsets of E, we say that measure μ defined on $B(E)$ is Haar measure if μ is regular measure and

$$\mu(sX) = \mu(X),\ \forall s \in E,\ \forall X \in B(E).$$

Definition 2. An object $\{E, S, \mu_i, i \in I\}$ is called Haar statistical structure, where $\{\mu_i, i \in I\}$ is a family of Haar probability measures on (E, S).

Definition 3. The Haar statistical structure $\{E, S, \mu_i, i \in I\}$ is called an orthogonal (singular) statistical structure if the family of probability measures $\{\mu_i, i \in I\}$ are pairwise singular measures.

Example 1. Let $E = [0,1]$ and S be the Borel σ- algebra of $[0,1]$. Let

$$\mu_1(B) = 2 \cdot l\left(B \cap \left[0, \frac{1}{2}\right]\right),\ B \in S,\ \mu_2(B) = 2 \cdot l\left(B \cap \left[\frac{1}{2}, 1\right]\right),\ B \in S,$$
$$\mu_3(B) = 2 \cdot l\left(B \cap \left[0, \frac{1}{3}\right]\right),\ B \in S,$$

where l is Lebesgue measure on S. Then $\mu_1 \perp \mu_2$ and $\mu_2 \perp \mu_3$, but μ_1 is not orthogonal to μ_3.

Definition 4. The Haar statistical structure $\{E, S, \mu_i, i \in I\}$ is called a weakly separable statistical structure if there exists a family S- measurable set $\{X_i, i \in I\}$ such that the relations are fulfilled:

$$\mu_i\left(X_{i'}\right) = \begin{cases} 1, & if \quad i = i'; \\ 0, & if \quad i \neq i'. \end{cases} \qquad \left(i, i' \in I\right).$$

Let $\{\mu_i, i \in I\}$ be Haar probability measures defined on the measurable spase (E,S). For each $i \in I$ denote by $\overline{\mu}_i$ the completion of the measure μ_i, and denote by $dom(\overline{\mu}_i)$ the σ- algebra of all $\overline{\mu}_i$-measurable subsets of E. Let $S_1 = \bigcap_{i\in I} dom(\overline{\mu}_i)$.

Definition 5. The Haar statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is called strongly separable statistical structure if there exist the family S_1-measurable sets $\{Z_i, i \in I\}$ such that the following relations are fulfilled:

1) $\overline{\mu}_i(Z_i) = 1,\ \forall i \in I;$
2) $Z_{i_1} \cap Z_{i_2} = \varnothing,\ \forall i_1 \neq i_2,\ i_1, i_2 \in I;$
3) $\bigcup_{i\in I} Z_i = E.$

Let I be the set of parameters and let $B(I)$ be σ- algebra of subsets of I which contains all finite subsets of I.

Definition 6. We will say that the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ admits a consistent estimate of parameters if there exists at least one measurable mapping

$$\delta : (E, S_1) \to (I, B(I)).$$

Such that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = 1,\ \ \forall i \in I.$$

Let M^σ be a real linear space of all alternating finite measures on S.

Definition 7. A linear subset $M_B \subset M^\sigma$ is called a Banach space of measures see [4] if:

1) The M_B one can introduce the norm so that M_B is a Banach space by this norm and also for only orthogonal measures $\mu, \nu \in M_B$ and a real number $\lambda \neq 0$ the equality $\|\mu + \lambda\nu\| \geq \mu$ is true;

2) if $\nu \in M_B$, f is a real measurable function with $|f(x)| \leq 1$ and $A \in S$ then

$$\nu_f(A) = \int_A f(x)\nu(dx) \in M_B$$

and $\|\nu_f\| \leq \|\mu\|$;

3) if $\nu_n \in M_B,\ \ \nu_n > 0,\ \ \nu_n(E) < \infty,\ \ n = 1,2,...$ and $\nu_n \downarrow 0,$ then for any linear functional $l^* \in M_B^*$:

$$\lim_{n\to\infty} l^*(\nu_n) = 0,$$

where M_B^* is a linear space congugate to M_B.

Definition 8. The Banach space $M_B = \left\{\{X_i\}_{i\in I};\ X_i \in M_{B_i},\ \forall i \in I,\ \sum_{i\in I}\|X_i\|_{M_{B_i}} < \infty\right\}$ with the norm $\|\{X_i\}_{i\in I}\| = \sum_{i\in I}\|X_i\|_{M_{B_i}}$ is called the direct sum of Banach space M_{B_i} and is denoted by

$$M_B = \bigoplus_{i\in I} M_{B_i}.$$

The following theorem was proved in [4].

Theorem 9. Let M_B be a Banach space of measures. Then there exists a family of pairwise orthogonal probability measures $\{\mu_i, i \in I\}$ from this space such that

$$M_B = \underset{i \in I}{\oplus} M_B(\mu_i),$$

where $M_B(\mu_i)$ is a Banach space of elements ν of the form

$$\nu B = \int_B f(x)\mu_i(dx),\ B \in S,\ \int_E |f(x)|\mu_i(dx) < \infty,$$

with the norm

$$\|\nu\|_{M_B(\mu_i)} = \int_E |f(x)|\mu_i(dx).$$

Remark 1. It is obvious that any Banach space of measures is a Banach space, whose elements are alternating measures, but not vice versa.

We define by $F = F(M_B)$ the set of real function f such that $\int_E f(x)\overline{\mu}_i(dx)$ is define for all $\overline{\mu}_i \in M_B$.

Theorem 10. Let

$$M_B = \underset{i \in I}{\oplus} M_B(\overline{\mu}_i)$$

be a Banach space of measure, $card\, I \le C$, Let E be a complete separable metric space, let $S_1 = \bigcap_{i \in I} dom(\overline{\mu}_i)$ is a Borel σ- algebra one E. In order for the Borel Haar orthogonal statistical structure $\{E, S_1, \mu_i, i \in I\}$ to admit a consistent estimate of parameters in the theory $(ZFC)\S(MA)$ it is necessary and sufficient that the correspondence $f \leftrightarrow \psi_f$ defined by the equality

$$\int_E f(x)\overline{\mu}_i(dx) = l(\overline{\mu}_i),\ \overline{\mu}_i \in M_B$$

was one-to- one (here l_f is a linear continuous functional on M_B, $f \in F(M_B)$).

Proof. Necessity, the existence of a consistent estimate of parameters $\delta : (E, S_1) \to (I, B(I))$ implies that $\overline{\mu}_i(\{x : \delta(x) = i\}) = 1,\ \forall i \in I$. Setting $X_i = (\{x : \delta(x) = i\}) = 1$, for $i \in I$ we get:

1) $\overline{\mu}_i(X_i) = 1,\ \forall i \in I;$
2) $X_{i_1} \cap X_{i_2} = \varnothing,$ for $\forall i_1 \ne i_2,\ i_1, i_2 \in I;$
3) $\bigcup_{i \in I} X_i = \{x : \delta(x) \in I\} = E.$

Therefore the Haar statistical structure $\{E, S_1, \mu_i, i \in I\}$ is strongly separable, hence there exist S_1-measurable sets $X_i\ (i \in I)$ such that

$$\overline{\mu}_i(X_{i'}) = \begin{cases} 1, & if \quad i = i'; \\ 0, & if \quad i \ne i'. \end{cases}$$

We put the linear continuous functional l_{c_i} into correspondence to a function $I_{c_i}(x) \in F_B(B)$

$$\int_E I_{c_i}(x)\overline{\mu}_i(dx) = l_{c_i}(\overline{\mu}_i) = \|\overline{\mu}_i\|_{M_B(\overline{\mu}_i)}.$$

Next we put the linear continuous functional $l_{f_{\psi_1}}$ into correspondence to be function $f_{\psi_1}(x) = f_1(x) \cdot I_{c_i}(x) \in F(M_B)$. Then for anly $\psi_2 \in M_B(\overline{\mu}_i)$:

$$\int_E f_{\psi_1}(x)\psi_2(dx) = \int_E f_{1_1}(x)I_{c_i}(x)\psi_2(dx) = \int_E f(x)f_1(x)I_{c_i}(x)\overline{\mu}_i(dx) =$$
$$= l_{f_{\psi_1}}(\psi_2) = \|\psi_2\|_{M_B(\overline{\mu}_i)}.$$

Let $\sum$ be the collection of extensions of functional l satisfying the condition $l_f \le p(x)$ on those subspace where they are defined, we introduce a partial ordering on $\sum$ an assuming $l_{f_1} < l_{f_2}$ if l_{f_2} is defined on a set larger than l_{f_1} and $l_{f_1}(x) = l_{f_2}(x)$, where they are both defined.

Let $\{l_{f_i}\}_{i\in I}$ be a linear ordered subset of $\sum$ and $M_B(\overline{\mu}_i)$ be the subspace on which l_{f_i} is defined. Define l_f on $\bigcup_{i\in I} M_B(\overline{\mu}_i)$ by assuming $l_f(x) = l_{f_i}(x)$ if $x \in M_B(\overline{\mu}_i)$. It is obvious that $l_{f_i} < l_f$. Since any linearly ordered subset $\sum$ has an upper bound, by Choru lemma $\sum$ contains a maximal element λ defined on the same set X' and satisfying the condition $\lambda(x) \le p(x)$ for $x \in X'$, but X' must coincide with the entire space M_B, otherwise, we could extend λ to a wider space by adding one more dimension. This contradicts the maximality of λ. Hence, $X' = M_B$. Therefore, the extension of the functional is defined everywhere. Let l_f be the lenear functional corresponding to the function $f(x) = \sum_{i\in I} g_i(x) I_{X_i}(x) \in F(M_B)$. Then we have

$$\int_E f(x)\overline{\mu}_i(dx) = \|\overline{\mu}\| = \sum_{i\in I} \|\overline{\mu}_i\|_{M_B(\overline{\mu}_i)},$$

where $\overline{\mu}(B) = \sum_{i\in I} \int_B g_i(x)\overline{\mu}_i(dx), \quad B \in S.$

Sufficiency. For $f \in F(M_B)$ we define a linear continuous functional l_f by $l_f(\nu) = \int_E f(x)\nu(dx)$. Denote by H_f a countable subset of I for which $\int_E f(x)\overline{\mu}_i(dx) = 0$ for $i \notin H_f$. Consider the functional l_{f_i} on $M_B(\overline{\mu}_i)$ corresponding to f_i. Then for $i_1, i_2 \in I$ we have:

$$\int_E f_{i_1}(x)\overline{\mu}_{i_2}(dx) = l_{f_{i_1}}(\overline{\mu}_{i_2}) = \int_E f_1(x) f_2(x)\overline{\mu}_i(dx) = \int_E f_{i_1}(x) f_2(x)\overline{\mu}_i(dx).$$

Therefore $f_{i_1} = f_1$ a.e. with respect to the measure $\overline{\mu}_i$.

Let $f_i(x) > 0$ a.e. with respect to the measure $\overline{\mu}_i$ and $\int_E f_i(x)\overline{\mu}_i(dx) < \infty$, $\overline{\mu}_i^*(c) = \int_c f_i(x)\overline{\mu}_i(dx)$, then $\int_E f_{\overline{\mu}_i^*}(x)\overline{\mu}_{i'}(dx) = 0, \ \forall i' \ne i$. Denote $C_i = \{x : f_{\overline{\mu}_i^*}(x) > 0\}$, then $\int_E f_{\overline{\mu}_i^*}(x)\overline{\mu}_{i'}(dx) = l_{\overline{\mu}_i^*}(\overline{\mu}_{h'}) = 0$, $\forall i' \ne i.$ Hence it follows that $\overline{\mu}_{i'}(C_i) = 0, \ \forall i' \ne i$.

On the other hand,

$$\overline{\mu}_i^*(E - C_i) = \int_{E-C_i} f_{\overline{\mu}_i}(x)\overline{\mu}_i(dx) = \int_E f_{\overline{\mu}_i}(x) I_{E-C_i}(x)\overline{\mu}_i(dx) = \int_E f_{\mu_i^*}(x) I_{E-C_i}(x)\overline{\mu}_i(dx) = 0$$

Since $f_{\overline{\mu}_i^*}(x) = f_{\overline{\mu}_i}(x)$ a.e. with respect to be measure $\overline{\mu}_{i'}$ and $f_{\mu_i^*}(x) I_{E-C_i}(x) \equiv 0$. Thus, the Haar statistical structure $\{E, S_1, \mu_i, i \in I\}$ is strongly separable.

Let us define an w_α-sequence of parts of the space E so that the following relations hold:

1) $(\forall i)(i < w_\alpha \Rightarrow B_i$ is a Borel subset of $E)$;

2) $(\forall i)(i < w_{\alpha} \Rightarrow B_i \subset X_i)$;

3) $(\forall i_1)(\forall i_2)(i_1 < w_{\alpha})\S(i_2 < w_{\alpha})\S(i_1 \neq i_2) \Rightarrow B_{i_1} \cap B_{i_2} = \varnothing$;

4) $(\forall i)(i < w_{\alpha} \Rightarrow \overline{\mu}_i(B_i) = 1)$.

Suppose $B_0 = X_0$. Let further the partial sequence $\{B_{i'}\}_{i'<i}$ is already defined for $i < w_{\alpha}$. It is clear that $\overline{\mu}_i^*\left(\bigcup_{i'<i} B_{i'}\right) = 0$. Thus there exists a Borel subset Y_i of the space E such that the following relations hold: $\bigcup_{i'<i} B_{i'} \subset Y_i$ and $\overline{\mu}_i(Y_i) = 0$. Suppose $B_i = X_i - Y_i$. Thus the w_{α}- sequence $\{B_{i'}\}_{i'<w_{\alpha}}$ of disjunctive measurable subset of space E constructed. Therefore $(\forall i)(i < w_{\alpha} \Rightarrow \overline{\mu}_i(B_i) = 1)$, where w_{α} denote the first ordinal number of the power of the set I. For $x \in E$ we put $\delta(x) = i$, where i is a parameter from the set I for which $x \in B_i$. Let now $Y \subset B(I)$. Then $\{x : \delta(x) \subset Y\} = \bigcup_{i \in Y} B_i$. We must show that $\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_i),\ \forall i \in I.$

If $i_0 \in Y$, then $\{x : \delta(x) \in Y\} = \bigcup_{i \in Y} B_i = B_{i_o} \bigcup\left(\bigcup_{i \in Y - i_o} B_i\right)$. It follows that $B_{i_0} \in S_1 \subseteq dom\overline{\mu}_{i_0}$. The validity of the condition $\bigcup_{i \in Y - \{i_o\}} B_i \subseteq \left(E - B_{i_0}\right)$, implies that $\overline{\mu}_{i_0}\left(\bigcup_{i \in Y - \{i_o\}} B_i\right) = 0$, the lust equality yields that $\bigcup_{i \in Y - i_o} B_i \in dom(\overline{\mu}_{i_0})$.

Since $dom\left(\overline{\mu}_{i_0}\right)$ is σ- algebra, we conclude that $\{x : \delta(x) \in Y\} = B_{i_o} \bigcup\left(\bigcup_{i \in Y - i_o} B_i\right) \in dom\left(\overline{\mu}_{i_0}\right)$. If $i_0 \notin Y$, then $\{x : \delta(x) \in Y\} = \bigcup_{i \in Y_o} B_i \subseteq (E - B_{i_0})$ and we conclude $\overline{\mu}_{i_0}\left(\{x : \delta(x) \in Y\}\right) = 0$. The last relation implies that $\{x : \delta(x) \in Y\} \in dom\left(\overline{\mu}_{i_0}\right)$. Thus we have shown the validity of the relation

$$\{x : \delta(x) \in Y\} \in dom\left(\overline{\mu}_{i_0}\right).$$

Thus we have shown the validity of the relation $\{x : \delta(x) \in Y\} \in dom\left(\overline{\mu}_{i_0}\right)$ for an arbitrary $i_0 \in I$, hence

$$\{x : \delta(x) \in Y\} \in \bigcap_{i \in I} dom\left(\overline{\mu}_i\right) = S_1.$$

We have shown the map $\delta : (E, S_1) \to (I, B(I))$ is a measurable map. Since $B(I)$ contains all singletons of I we conclude that $\overline{\mu}_i\{x : \delta(x) \in i\} = \overline{\mu}_i(B_i) = 1,\ \forall i \in I.$

მათემატიკა

ჰაარის სტატისტიკური სტრუქტურების პარამეტრთა ძალდებული შეფასებები ბანახის ზომათა სივრცეში

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ალბათური მახასიათებლების დასადგენად გამოყენებულ უნდა იქნეს სტატისტიკური მეთოდები. სტატისტიკურ პრობლემებს შორის არის კლასი, რომელშიც დაკვირვებების რაოდენობა ერთადერთია. მიუხედავად ამისა, ბევრ შემთხვევაში პრობლემაა უცნობის განაწილების პარამეტრების მნიშვნელობების საიმედოდ განსაზღვრა ან საიმედოდ არჩევა ერთ-ერთი ჰიპოთეზის განაწილების ფორმის შესახებ უსასრულო რაოდენობის ალტერნატიულ ჰიპოთეზებს შორის. იმ შემთხვევაში, როდესაც პარამეტრის მნიშვნელობა საიმედოდ განისაზღვრება, ერთი დაკვირვებით ვიტყვით, რომ მისთვის გვაქვს პარამეტრის ძალდებული შეფასება. აღნიშნულ ნაშრომში ჩვენ განვსაზღვრეთ ჰაარის სტატისტიკური სტრუქტურა. დამტკიცებულია ბანახის სივრცეში ჰაარის სტატისტიკური სტრუქტურის პარამეტრების ძალდებული შეფასებების არსებობის აუცილებელი და საკმარისი პირობები.

REFERENCES

1. Jech T. (2003) Set Theory. Springer-Verlag, Berlin.
2. Ibramkhalilov I. and Skorokhod A. (1980) Sostoiatel'nye otsenki parametrov sluchainykh protsessov (Consistent estimates of parameters of random processes), *Naukova Dumka*, Kiev (in Russian).
3. Zerakidze Z., Purtukhia 0. (2021) The consistent criteria for hypothesis testing. *Georgian Mathematical Journal,* **28,** 5: 787-791.
4. Zerakidze Z., Purtukhia 0. (2017) The weakly consistent, strongly consistent and consistent estimator of the parameters, Reports of Enlarged of the I. Vekua Institute of Applied Mathematics, 31: 151-154.
5. Zerakidze Z. and Patsatsia M. (2018) Extreme points and consistent criteria for hypothesis testing in a Banach space, *Bull. Georg. Natl. Acad. Sci.*, **12**(4): 33-39.

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