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Mathematics

Consistent Estimators of Parameters and Consistent Hypotheses Testing Criteria for the Haar Statistical Structures

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Statistics of random processes is used in various fields of science and technology. When using random processes as models of real phenomena, the question arises of determining the probabilistic characteristics of the process. Statistical method should be used to determine probabilistic characteristics. Among the problems of statistics, there is a class of problems in which the number of observation is unique. Despite the uniqueness of observations in many cases it is possible to reliably determine the values of unknown distribution parameters or to reliably choose one of infinite numbers of competing hypotheses about the exact form of the distribution. In the case when parameter or hypotheses are reliably determined by one observation, it is said that there exists a consistent estimate of parameter. In this paper, we define Haar statistical structure. Necessary and sufficient conditions are given both for the existence of consistent estimators of parameters and a consistent criterion for hypotheses testing. *© 2023 Bull. Georg. Natl. Acad. Sci.*

consistent estimator, consistent criterion, Haar statistical structure, strongly separable statistical structure

Let (E,S) be a measurable space with given family of probability measures $\{\mu_h, h \in H\}$. An object $\{E,S,\mu_h, h \in H\}$ is called a statistical structure.

By (ZFC) we denote the formal system of Zermelo-Fraenkel with the addition of axiom of choice (AC), i.e. (ZFC) = (ZF) & (AC). By (ZFC) & (CH) we denote the theory with the addition of a continuum hypothesis (CH): $2^{\aleph_0} = \omega_1$, where ω_1 denotes the first uncountable cardinal number, and by (ZFC) & (MA) we denote the theory with the addition of Martin's axiom (MA). It is known that in the theory (ZFC) & (CH) Martin's axiom (MA) is automatically satisfied. It is well known that Martin's axiom (MA) is much weaker than the continuum hypothesis (CH). Moreover, the negation of the continuum hypothesis ($\neg$ CH) is compatible with Martin's axiom [1].

It is known that if a statistical structure admits weakly consistent, consistent, or strongly consistent estimators, then this statistical structure is orthogonal [2]. The notions and corresponding constructions of weak separability and strong separability were introduced and studied by A. Skorokhod [3]. The notion and corresponding construction of consistent criteria for hypothesis testing were introduced and studied by Z. Zerakidze. It is known that if a statistical structure admits either consistent estimators or consistent criterion for hypotheses testing then this statistical structure is strongly separable but not vice versa [4]. Z. Zerakidze and O. Purtukhia constructed a strongly separable statistical structure that does not admit a consistent criterion for hypotheses testing in the theory (ZF) [5]. A. Skorokhod proved that in the theory (ZFC) & (CH) an arbitrary weakly separable statistical structure, whose cardinality is not greater than the cardinality of the continuum, is strongly separable [6]. Z. Zerakidze proved that in the theory (ZFC) & (MA) Borel weakly separable statistical structure, whose cardinality is not greater than the cardinality of the continuum, is strongly separable [7].

This paper is devoted to the question of the existence of consistent estimators of parameters and the existence of consistent criteria for hypotheses testing for the Haar statistical structure.

We recall some definitions from the works [8].

Definition 1. Let E be an arbitrary locally compact and σ -compact topological group and $B(E)$ is σ -algebra of subsets of E. We say that measure μ defined on $B(E)$ is Haar measure if μ is regular measure and

$$\mu(sX) = \mu(X), \quad \forall s \in E, \quad \forall X \in B(E).$$

Definition 2. An object $\{E, S, \mu_h, h \in H\}$ is called Haar statistical structure, where $\{\mu_h, h \in H\}$ is a family of Haar probability measures on $(E, B(E))$.

Definition 3. A Haar statistical structure $\{E, S, \mu_h, h \in H\}$ is called orthogonal (singular) Haar statistical structure if the family of probability measures $\{\mu_h, h \in H\}$ are pairwise orthogonal measures.

Definition 4. A Haar statistical structure $\{E, S, \mu_h, h \in H\}$ is called weakly separable statistical structure if there exists a family S –measurable sets $\{X_h, h \in H\}$ such that the relations are fulfilled:

$$\mu_h(X_{h'}) = \begin{cases} 1, & if \ h = h'; \\ 0, & if \ h \neq h' \end{cases} \qquad (h, h' \in H).$$

Definition 5. A Haar statistical structure $\{E, S, \mu_h, h \in H\}$ is called separable statistical structure if there exists a family of S -measurable sets $\{X_h, h \in H\}$ such that the following relations are fulfilled:

$$\mu_h(X_{h'}) = \begin{cases} 1, & if \ h = h'; \\ 0, & if \ h \neq h' \end{cases} \qquad (h, h' \in H);$$

$$\forall h, h' \in H : card(X_h \cap X_{h'}) < c, \ if \quad h \neq h',$$

where c denotes the cardinality of the continuum.

Definition 6. A Haar statistical structure $\{E, S, \mu_h, h \in H\}$ is called strongly separable statistical structure if there exists a disjoint family S – measurable sets $\{X_h, h \in H\}$ such that

$$\mu_h(X_h) = 1, \ \forall h \in H.$$

Remark 1. Strong separability implies separability, separability implies weak separability, weak separability implies orthogonality, but not vice versa [9].

Remark 2. On an arbitrary continuum power E of a locally compact and σ-compact topological group, one can define an orthogonal Haar statistical structure with the maximum possible cardinality equal to 2^{2^c}, a weakly separable Haar statistical structure with the maximum possible cardinality equals 2^c and a strongly separable Haar statistical structure with the maximum possible cardinality equals c [10].

Let H be the set of hypotheses and let $B(H)$ be σ-algebra of subsets of H which contains all finite subsets of H.

Definition 7. A statistical criterion for hypotheses testing is any measurable mapping:

$$\delta : (E, S) \to (H, B(H)).$$

Definition 8. We will say that the Haar statistical structure $\{E, S, \mu_h, h \in H\}$ admits a consistent criterion for hypothesis testing if there exists at least one measurable mapping $\delta : (E, S) \to (H, B(H)),$ such that

$$\mu_h\left(\{x : \delta(x) = h\}\right) = 1, \ \forall\, h \in H.$$

Let I be the set of parameters and let $B(I)$ be σ-algebra of subsets of I which contains all finite subsets of I.

Definition 9. We will say that the Haar statistical structure $\{E, S, \mu_i, i \in I\}$ admits a consistent estimators of parameters $i \in I$ if there exists at least one measurable mapping $\delta : (E, S) \to (I, B(I)),$ such that

$$\mu_i\left(\{x : \delta(x) = i\}\right) = 1, \ \forall\, i \in I.$$

Remark 3. There are statistical structures that admit a consistent estimators of parameters but do not admit a consistent criterion for hypothesis testing [11].

Theorem 1. In (ZF) theory, the Haar statistical structure $\{E, S, \mu_i, i \in N\}$ (where N is the set of natural numbers) admits a consistent estimators of parameters $i \in N$ if and only if when it is either strongly separable, or separable, or weakly separable, or orthogonal.

Proof. Necessity. Since Haar statistical structure $\{E, S, \mu_i, i \in N\}$ admits a consistent estimators of parameters $i \in N$, there exists a measurable mapping $\delta : (E, S) \to (N, B(N)),$ such that $\mu_i(\{x : \delta(x) = i\}) = 1,$ $\forall\, i \in N.$ Let $X_i = \{x : \delta(x) = i\},$ then it is evident that $X_{i'} \cap X_{i''} = \varnothing, \ \forall i' \neq i''$ and $\mu_i(X_i) = 1, \ \forall i \in N.$ Therefore, Haar statistical structure is strongly separable. Hence, according to the Remark 1 the proof of necessity is completed.

Sufficiency. Due to the orthogonality of Haar statistical structure $\{E,S,\mu_h,h\in N\}$ there exists the family of S -measurable sets $\{X_{h,h^i}\}$ such that for any $h\neq h^{'}$ $\mu_{h^{'}}(X_{h,h^i})=0$ and $\mu_h(E\setminus X_{h,h^i})=0.$ Therefore, if we consider the sets $X_h=\underset{h^{'}\neq h}{\cup}(E\setminus X_{h,h^i}),$ we get $\mu_h(X_h)=0.$ Hence, $\mu_h(E\setminus X_h)=1.$ On the other hand, for $h\neq h^{'}$ we have $\mu_{h^{'}}(E\setminus X_h)=0.$ It means that statistical structure $\{E,S,\mu_h,h\in N\}$ is weakly separable [12]. Therefore, there exists the family of S -measurable sets $\{X_h,h\in N\}$ such that

$$\mu_h(X_{h^{'}})=\begin{cases}1, & if\ h=h^{'};\\ 0, & if\ h\neq h^{'}.\end{cases}$$

Consider now the sets $\overline{X}_h=X_h\setminus(X\cap(\underset{h^{'}\neq h}{\cup}X_{h^{'}})),$ $h\in N.$ It is obvious that these sets are S -measurable disjoint sets and $\mu_h(\overline{X}_h)=1,$ $\forall h\in N.$ Let us define the mapping $\delta:(E,S)\to(H,B(H))$ in the following way $\delta(\overline{X}_i)=1,$ $\forall i\in N.$ Then we have $\{x:\delta(x)=i\}=\overline{X}_i$ and $\mu_i(\{x:\delta(x)=i\})=1,$ $\forall i\in N.$ Hence, δ is a consistent criterion for hypothesis testing. Further, it follows from the existence of consistent estimators of parameters that this structure is strongly separable, and orthogonality follows from strong separability.

The following theorem is proved in a similar way.

Theorem 2. In (ZF) theory, the Haar statistical structure $\{E,S,\mu_h,h\in N\}$ admits a consistent criteria for hypothesis testing if and only if when it is either strongly separable, or separable, or weakly separable, or orthogonal.

Let $\{\mu_h,h\in H\}$ be probability measures defined on the measurable space (E,S). For each $h\in H$ denote by $\overline{\mu}_h$ the completion of the measure μ_h, and denote by $dom(\overline{\mu}_h)$ the σ -algebra of all $\overline{\mu}_h$ -measurable subsets of E . Let

$$S_1=\cap_{h\in H}dom(\overline{\mu}_h).$$

Definition 10. The Haar statistical structure $\{E,S_1,\overline{\mu}_h,h\in N\}$ is called strongly separable if there exists the family of S_1 -measurable sets $\{Z_h,h\in H\}$ such that the relations are fulfilled:

$$1)\ \overline{\mu}_h(Z_h)=1,\ \forall h\in H;$$

$$2)\ Z_{h_1}\cap Z_{h_2}=\varnothing,\ \forall h_1\neq h_2,\ h_1,h_2\in H;$$

$$3)\ \underset{h\in H}{\cup}Z_h=E.$$

Definition 11. We will say that the orthogonal Haar statistical structure $\{E,S_1,\overline{\mu}_h,h\in H\}$ admits a consistent criterion for hypothesis testing if there exists at least one measurable mapping $\delta:(E,S_1)\to(H,B(H)),$ such that

$$\overline{\mu}_h(\{x:\delta(x)=h\})=1,\ \forall\ h\in H.$$

Definition 12. We will say that the orthogonal Haar statistical structure $\{E,S_1,\overline{\mu}_i,i\in I\}$ admits a consistent estimators of parameters $i\in I$ if there exists at least one measurable mapping $\delta:(E,S_1)\to(I,B(I)),$ such that

$$\overline{\mu}_i(\{x:\delta(x)=i\})=1,\ \forall\ i\in I.$$

Theorem 3. In order that the orthogonal Haar statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$, $cardH = c,$ to be a consistent criterion for hypotheses testing in the theory of $(ZFC)\,\&\,(CH)$ it is necessary and sufficient that this statistical structure be strongly separable (definition 10).

Proof. Necessity. The existence of a consistent criterion for hypotheses testing $\delta:(E,S_1)\to(H,B(H))$ implies that $\overline{\mu}_h(\{x:\delta(x)=h\})=1,\ \forall h\in H.$ Setting $X_h=\{x:\delta(x)=h\}$ for $h\in H$ we get:

$$1)\ \overline{\mu}_h(X_h)=1,\ \forall h\in H;$$

$$2)\ X_{h'}\cap X_{h''}=\varnothing,\ \forall h'\neq h'';\ h',h''\in H;$$

$$3)\ \bigcup_{h\in H} X_h = E.$$

Hence, the statistical structure $\{E,S_1,\overline{\mu}_h,h\in H\}$ is strongly separable.

Sufficiency. Since Haar statistical structure $\{E,S_1,\overline{\mu}_h,h\in H\}$, $cardH=c,$ is strongly separable, there exists a family $\{Z_h,h\in H\}$ of elements of the σ-algebra $S_1=\cap_{h\in H}dom(\overline{\mu}_h)$ such that:

$$1)\ \overline{\mu}_h(Z_h)=1,\ \forall h\in H;$$

$$2)\ Z_{h_1}\cap Z_{h_2}=\varnothing,\ \forall h_1\neq h_2;\ h_1,h_2\in H;$$

$$3)\ \bigcup_{h\in H} Z_h = E.$$

For $x\in E,$ we put $\delta(x)=h,$ where h is the unique hypothesis from the set H for which $x\in Z_h$. The existence and uniqueness of such a hypothesis h from H can be proved using conditions 2), 3).

Take now $Y\in B(H),$ then $\{x:\delta(x)\in Y\}=\cup_{h\in Y}Z_h.$ We must show that $\{x:\delta(x)\in Y\}\in dom(\overline{\mu}_h)$ for each $h\in H$.

If $h_0\in Y,$ then

$$\{x:\delta(x)\in Y\}=\cup_{h\in Y}Z_h = Z_{h_0}\cup(\cup_{h\in Y\setminus\{h_0\}}Z_h).$$

On the one hand, from the conditions 1), 2), 3) it follows that

$$Z_{h_0}\in S_1=\cap_{h\in H}dom(\overline{\mu}_h)\subseteq dom(\overline{\mu}_{h_0}).\ \$$$

On the other hand, the inclusion

$$\cup_{h\in Y\setminus\{h_0\}}Z_h\subseteq(E\setminus Z_{h_0})$$

implies that

$$\overline{\mu}_{h_0}(\cup_{h\in Y\setminus\{h_0\}}Z_h)=0$$

and hence,

$$\cup_{h\in Y\setminus\{h_0\}}Z_h\in dom(\overline{\mu}_{h_0}).$$

Since $dom(\overline{\mu}_{h_0})$ is a σ-algebra, we conclude that

$\{x:\delta(x)\in Y\}=Z_{h_0}\cup(\cup_{h\in Y\setminus\{h_0\}}Z_h)\in dom(\overline{\mu}_{h_0}).$

If $h_0 \notin Y,$ then $\{x:\delta(x)\in Y\}=\cup_{h\in Y} Z_h \subseteq (E \setminus Z_{h_0})$ and we obtain that $\overline{\mu}_{h_0}\{x:\delta(x)\in Y\}=0.$ The last relation implies that

$$\{x:\delta(x)\in Y\}\in dom(\overline{\mu}_{h_0}),\ \ \forall Y\in B(H).$$

Thus, we proved the validity of the relation

$$\{x:\delta(x)\in Y\}\in dom(\overline{\mu}_{h_0})$$

for any arbitrary $h_0 \in H.$ Hence,

$$\{x:\delta(x)\in Y\}\in \bigcap_{h\in H} dom(\overline{\mu}_h)=S_1.$$

Therefore, the mapping $\delta:(E,S_1)\to(H,B(H))$ is a measurable mapping.

Since $B(H)$ contains all finite subsets of H , we ascertain that

$$\overline{\mu}_h(\{x:\delta(x)=h\})=\overline{\mu}_h(Z_h)=1,\ \ \forall h\in H,$$

i. e. this statistical structure admits a consistent criterion for hypotheses testing.

The following theorem is proved similarly.

Theorem 4. In order that the orthogonal Haar statistical structure $\{E,S_1,\ \overline{\mu}_h,h\in H\}$, $cardH=c,$ to admit a consistent estimators of parameters in the theory of $(ZFC)\,\&\,(CH)$ it is necessary and sufficient that this statistical structure be strongly separable (definition 10).

მათემატიკა

ჰაარის სტატისტიკური სტრუქტურების პარამეტრების ძალდებული შეფასებები და ჰიპოთეზების შემოწმების ძალდებული კრიტერიუმები

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** გორის სახელმწიფო სასწავლო უნივერსიტეტი, განათლების, ზუსტ და საბუნებისმეტყველო მეცნიერებათა ფაკულტეტი, გორი, საქართველო*

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

შემთხვევითი პროცესების სტატისტიკა გამოიყენება მეცნიერებისა და ტექნოლოგიების სხვადასხვა დარგში. შემთხვევითი პროცესების რეალური მოვლენების მოდელებად გამოყენებისას წარმოიქმნება საკითხი პროცესების ალბათური მახასიათებლების განსაზღვრის შესახებ. სავარაუდო მახასიათებლების დასადგენად უნდა იქნეს გამოყენებული სტატისტიკური მეთოდი. სტატისტიკის პრობლემებს შორის არის პრობლემათა კლასი, რომლებშიც დაკვირვების რაოდენობა ერთადერთია. მიუხედავად დაკვირვების ერთადერთობისა, ხშირ შემთხვევაში შესაძლებელია საიმედოდ განისაზღვროს უცნობი განაწილების პარამეტრების მნიშვნელობები ან საიმედოდ შეირჩეს ერთი ჰიპოთეზა განაწილების ზუსტი ფორმის შესახებ კონკურენტ ჰიპოთეზათა უსასრულო რაოდენობიდან. იმ შემთხვევაში, როდესაც პარამეტრი ან ჰიპოთეზები საიმედოდ განისაზღვრება ერთი დაკვირვებით, მაშინ ამბობენ, რომ არსებობს პარამეტრის ძალდებული შეფასება. წარმოდგენილ ნაშრომში განვსაზღვრავთ ჰაარის სტატისტიკურ სტრუქტურას. მოცემულია აუცილებელი და საკმარისი პირობები როგორც პარამეტრების ძალდებული შემფასებლების არსებობისთვის, ასევე ჰიპოთეზების ტესტირების ძალდებული კრიტერიუმისთვის.

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