ᲡᲐᲥᲐᲠᲗᲕᲔᲚᲝᲡ ᲛᲔᲪᲜᲘᲔᲠᲔᲑᲐᲗᲐ ᲔᲠᲝᲕᲜᲣᲚᲘ ᲐᲙᲐᲓᲔᲛᲘᲘᲡ ᲛᲝᲐᲛᲑᲔ, ტ. 17, №4, 2023
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 17, no. 4, 2023

Mathematics

On the Asymptotic Estimation for Generalized Dirichlet Integrals

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The studies of Kolmogorov and Nikolsky laid the basis for the study of the asymptotics of trigonometric Fourier series of continuous functions. In this direction significant results were obtained by Stepanets who studied the asymptotic behavior of trigonometric Fourier series in the class of continuous functions. In this work, continuing the relevant studies of Stepanets and Taberski, an asymptotic estimation for generalized Dirichlet integrals is obtained. In particular, under specific circumstances our result coincides with theorem of A. Stepanets. © *2023 Bull. Georg. Natl. Acad. Sci.*

trigonometric Fourier series, general Dirichlet integrals, modulus of continuity

Let f be a 2π-periodic locally integrable function and let

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

be the partial sums of Fourier series of f with respect to the trigonometric system (see [1]). If f is a continuous function on $[a,b]$ then

$$\omega(\delta, f) = \sup\left\{ \left| f(x_1) - f(x_2) \right| : \left| x_1 - x_2 \right| \le \delta,\ \ x_1, x_2 \in [a,b] \right\}$$

is called the modulus of continuity of f. For a modulus of continuity ω denote by $H_\omega[a,b]$ class of functions f with property $\left| f(x) - f(x') \right| \le \omega(\left| x - x' \right|)$, $x, x' \in [a,b]$. Korneichuk [2] proved the statement, which in the sequel was named as Korneichuk-Stechkin lemma. In particular, Korneichuk received the estimation of the following value

$$\mathcal{E}_\omega(\psi) \overset{def}{=} \sup_{f \in H_\omega[a,b]} \left| \int_a^b f(t)\psi(t)dt \right|, \tag{1}$$

where ψ is an integrable function with the average mean 0 on $[a,b]$. In addition sign of ψ on $[a,c]$ and $[c,b]$, $a<c<b$, maintains almost everywhere (in this case we write $\psi \in V_{a,b}^{c}$). The estimation of (1) for a convex modulus of continuity ω is exact and explicitly is given. In this work, we use Korneichuk-Stechkin lemma in the following form.

Lemma 1. (Korneichuk-Stechkin) Let ω be any modulus of continuity, $\psi(t) \in V_{a,b}^{c}$, $c=\dfrac{a+b}{2}$ and $\psi(t)=-\psi(2c-t)$, then

$$\mathcal{E}_{\omega}(\psi) \le \int_{a}^{c} |\psi(t)| \omega\big(2(c-t)\big) dt = \int_{c}^{b} |\psi(t)| \omega\big(2(t-c)\big) dt. \tag{2}$$

If modulus of continuity ω is convex, then the equality in (2) is achieved for function from $K \pm f_{*}(x)$, where K is a constant and

$$f_{*}(x)=\begin{cases} -\dfrac{1}{2}\omega\big(2(c-x)\big), x\in[a,c], \\ \dfrac{1}{2}\omega\big(2(c-x)\big), x\in[c,b]. \end{cases}$$

Based on the above Korneichuk-Stechkin lemma, Stepanets [3] proved many statements. We present one of them.

Theorem 1. (Stepanets) For any ω modulus of continuity we have asymptotic inequality

$$\mathcal{E}\big(H_{\omega};S_{n}\big) \le \frac{2}{\pi^{2}} \ln n \int_{0}^{\pi/2} \omega\left(\frac{4t}{2n+1}\right) \sin t\, dt + O\left(\omega\left(\frac{1}{n}\right)\right), \tag{3}$$

where $\mathcal{E}\big(H_{\omega};S_{n}\big)=\sup\limits_{f\in H_{\omega}} \left\| S_{n}(f;x)-f(x) \right\|_{C}$. In the case ω is a convex continuous function, (3) becomes the equality.

Taberski [4] considered the following quantities:

$$a_{k}^{l}=\frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{k\pi t}{l}dt, \quad b_{k}^{l}=\frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{k\pi t}{l}dt,$$

$$S_{n}^{l}(x,f)=\frac{a_{0}}{2}+\sum_{k=1}^{n} a_{k}^{l}\cos kx+b_{k}^{l}\sin kx,$$

where f is a locally integrable function on $x\in(-\infty,\infty)$, $l>0$, and $n=1,2,3,\dots$. The last sum can be represented by Dirichlet integral as follows:

$$S_{n}^{l}(x,f)=\frac{1}{l}\int_{-l}^{l} f(u)D_{n}^{l}(u-x)du,$$

where

$$D_{n}^{l}(t)=\frac{1}{2}+\sum_{k=1}^{n}\cos\frac{k\pi t}{l}=\frac{\sin(2n+1)\pi t/2l}{2\sin \pi t/2l}.$$

If f is a locally integrable periodic function with period $2\pi,$ then the last integral for $l=\pi$ coincide with the partial sums of the trigonometric Fourier series.

Definition 1. Let f be an uniformly continuous function on $\mathbb{R}$. We say $f\in H_\omega$ if for each $t_1,t_2\in\mathbb{R}$

$$\left|f(t_1)-f(t_2)\right|\le\omega(|\,t_1-t_2\,|)\,,$$

where ω is a modulus of continuity.

Let

$$\mathcal{E}\left(H_\omega;S_n^l\right)\overset{def}{=}\sup_{f\in H_\omega}\sup_{x\in\mathbb{R}}\left|S_n^l(f;x)-f(x)\right|=\sup_{f\in H_\omega}\left\|\frac{1}{l}\int_{-l-x}^{l-x}[f(x+t)-f(x)]D_n^l(t)dt\right\|_C .$$

Theorem 2. For any modulus of continuity ω

$$\mathcal{E}\left(H_\omega,S_n^l\right)\le\frac{2}{\pi^2}\ln n\int_0^{\pi/2}\omega\left(\frac{4lt}{(2n+1)\pi}\right)\sin t\,dt+O\left(\omega\left(\frac{l}{n}\right)\right).$$

If ω is a convex continuous function, then in the last inequality the equality holds.

Corollary 1. For any modulus of continuity ω we have the asymptotic equality:

$$\mathcal{E}\left(H_\omega,S_n^l\right)=\frac{2s_n^l(\omega)}{\pi^2}\ln n+O\left(\omega\left(\frac{l}{n}\right)\right),$$

where

$$s_n^l(\omega)=\sup_{f\in H_\omega\left[-\frac{l}{2n+1};\frac{l}{2n+1}\right]}\left|\int_{-\pi/2}^{\pi/2}f\left(\frac{2lt}{(2n+1)\pi}\right)\sin t\,dt\right|.$$

Corollary 2. For any modulus of continuity $\omega=\omega(t)$

$$\mathcal{E}\left(H_\omega,S_n^l\right)=\Theta_\omega^l\frac{2}{\pi^2}\ln n\int_0^{\pi/2}\omega\left(\frac{4lt}{(2n+1)\pi}\right)\sin t\,dt+O\left(\omega\left(\frac{l}{n}\right)\right),$$

where

$$2/3\le\Theta_\omega^l\le 1.$$

Moreover, $\Theta_\omega^l=1$ if $\omega(t)$ is a convex function.

მათემატიკა

განზოგადებული დირიხლეს ინტეგრალების ასიმპტოტური შეფასების შესახებ

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კოლმოგოროვისა და ნიკოლსკის გამოკვლევებმა საფუძველი ჩაუყარეს უწყვეტი ფუნქციების ფურიეს ტრიგონომეტრიული მწკრივების ასიმპტოტური ყოფაქცევის შესწავლას. ამ მიმართულებით მნიშვნელოვანი შედეგები მიიღო სტეპანეცმა, რომელმაც შეისწავლა უწყვეტ ფუნქციათა კლასში ფურიეს ტრიგონომეტრიული მწკრივების ასიმპტოტური ყოფაქცევა. ნაშრომში, გრძელდება რა სტეპანეცისა და ტაბერსკის კვლევები, მიღებულია განზოგადებული დირიხლეს ინტეგრალების ასიმპტოტური შეფასება. მიღებული შედეგიდან, როგორც ერთ-ერთი კერძო შემთხვევა, გამომდინარეობს სტეპანეცის თეორემა.

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Received November, 2023