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Gaver's Parallel System with Repair and Redundancy Delay

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Having used the method of supplementary variable in conjunction with purely probabilistic reasoning Gaver's parallel system with repair and redundancy delay is investigated. This method results in significant generalization of classical parallel systems. New approach to investigation allows us to forgo solving partial differential equations (Kolmogorov forward equations) of non-classical boundary value problem of mathematical physics with non-local boundary conditions and directly derive the system's solution in terms of Laplace transforms. *© 2023 Bull. Georg. Natl. Acad. Sci.*

semi-Markov process, probabilistic reasoning, Laplace transform, failure, redundancy, repair, redundancy delay

New method introduced in [1-3] is applied to model describing long-run availability of parallel systems consisting of two units. The proposed pure probabilistic approach significantly simplifies derivation of Laplace transforms of probability distribution functions, which are of primary interest.

We would like to pay due respect to foundational work by D.P. Gaver, JR., showing transient solution to the system and hopefully making his results more easily accessible to the readers who take interest in semi-Markov processes.

Description of Gaver's Parallel System

In this section, we introduce two-unit system as introduced by D.P. Gaver, JR in [4], commonly called Gaver's Parallel System [5,6]. Only slight alterations are made to original notation to put it in accordance with modern notation.

The repair (service) time of the elements is independent and identically distributed random variables having distribution function G and probability density function g. The service rate function is denoted by μ.

Here we take into account one more influencing factor, which is often very important for the model to be adequate to real-world problems. That is the following: the redundant unit can be not available for replacement of failed main unit (at the moment the redundant one is in delay).

It is supposed that delays occur as Poisson flow with intensity λ_1 and the duration of the delay is a random variable with exponential distribution. The distribution parameter is γ_1.

The next random processes give the state of the system at time t:

n(t) – number of failed elements at time t;

ξ(t) – elapsed repair time when an element is in a service at time t.

The system state space can be represented as $E=\{0,0d,1,1d,2\}$

0 – both main and redundant units are operative; the redundant unit is available (is not in delay);

$0d$ – both main and redundant units are operative: the redundant unit is not available (is in delay);

1 – one unit is failed and is under repair: the second one is operative and functions as main;

$1d$ – one unit is failed and is under repair: the second one is not available (is in delay);

2 – both main and redundant units are failed (the first one is under repair).

We introduce the following functions as probabilistic characteristics of the considered system:

$P_0(t), P_0^d(t), P_1(x,t), P_1^d(x,t), P_2(x,t)$,

$P_0(t) = \mathbb{p}\{n(t) = 0;$ the redundant unit is available$\}$,

$P_0^d(t) = \mathbb{p}\{n(t) = 0;$ the redundant unit is not available$\}$,

$P_1(x,t) = \lim_{h\to 0}\frac{1}{h}\mathbb{p}\{n(t) = 1; x < \xi(t) < x+h;$ the operative unit functions as main one$\}$,

$P_1^d(x,t) = \lim_{h\to 0}\frac{1}{h}\mathbb{p}\{n(t) = 1; x < \xi(t) < x+h;$ the second unit is in delay$\}$,

$P_2(x,t) = \lim_{h\to 0}\frac{1}{h}\mathbb{p}\{n(t) = 2; x < \xi(t) < x+h\}$.

Using standard probabilistic reasonings, the nonclassical problem of mathematical physics with non-local boundary conditions can be derived:

$$\frac{d}{dt}P_0(t) = -(\alpha+\beta)P_0(t) + \gamma_1 P_0^d(t) + \int_0^t P_1(x,t)\mu(x)dx, \tag{1}$$

$$\frac{d}{dt}P_0^d(t) = -(\alpha+\beta+\gamma)P_0^d(t) + \lambda P_0(t) + \int_0^t P_1^d(x,t)\mu(x)dx, \tag{2}$$

$$\frac{\partial}{\partial t}P_1(x,t) + \frac{\partial}{\partial t}P_1(x,t) = -[\alpha+\mu(x)]P_1(x,t) + \gamma P_1^d(x,t), \tag{3}$$

$$\frac{\partial}{\partial t}P_1^d(x,t) + \frac{\partial}{\partial t}P_1(x,t) = -[\beta+\mu(x)]P_1^d(x,t), \tag{4}$$

$$\frac{\partial}{\partial t}P_2(x,t) + \frac{\partial}{\partial t}P_2(x,t) = -\mu(x)P_2(x,t) + \alpha P_1(x,t) + \gamma P_1^d(x,t), \tag{5}$$

$$P_1(0,t) = (\alpha+\beta)P_0(t) + \gamma P_0^d(t) + \int_0^t P_2(x,t)\mu(x)dx, \tag{6}$$

$$P_1^d(0,t) = 0, \tag{7}$$

$$P_2(0,t) = 0. \tag{8}$$

The following initial conditions are supposed

$$P_0(0) = 1;\ P_0^d(0) = 0\ ;\ P_1(x,0) = 0;\ P_1^d(x,0) = 0\ ;\ P_2(x,0) = 0. \tag{9}$$

According to the same probabilistic reasoning as in [1-3] the following Theorem has been proved.

Theorem. $P_1(x,t)$, $P_1^d(x,t)$ and $P_2(x,t)$ can be expressed as:

$$P_1(x,t) = P_1(0,t-x)e^{-\alpha x}[1-G(x)] + P_1^d(0,t-x)[1-G(x)]\int_0^x e^{-\alpha y}e^{-\alpha(x-y)}dy, \tag{10}$$

$$P_1^d(x,t) = P_1^d(0,t-x)[1-G(x)]e^{-(\beta+\gamma)x}, \tag{11}$$

$$P_2(x,t) = P_1(0,t-x)[1-G(x)](1-e^{-\alpha x}) + P_1^d(0,t-x)[1-G(x)]\left(1-e^{\beta x}\right). \tag{12}$$

Using (10), (11) and (12) from (1), (2), (6) and (8) Laplace transforms for all probabilistic characteristics introduced above can be derived. In other words, it can be derived transient and steady state solutions of the considered system without use of partial-differential equations (Kolmogorov equations) in above constructed boundary value problem (BVP). This surprisingly important fact is the result of using new method discussed in [1-3]. The latter is the further development of Cox method [7] and it can be called modified supplementary variables method (SVM) or SVM without partial differential equations. We hope this approach will attract more attention of leading experts in the field.

Conclusions and Remarks

There is oftentimes a "hidden", simple and pure probabilistic approach to investigating the stochastic systems.

It is often illustrated in the publications of the most prominent mathematicians: A.N. Kolmogorov, B.V. Gnedenko, W. Feller, and famous experts in queuing theory: L. Kleinrock, R.B. Cooper, etc. They strongly recommend exerting effort to discover this "hidden" approach [1-3].

We had good fortune in this paper and in [1-3] to capitalize on such an opportunity and to take queueing and reliability in new, unexpected and very interesting direction.

Remember, once again that the supplementary variables method in MTR and QT leads to a non-classical BVP of Mathematical Physics with nonlocal boundary conditions. The main parts of the BVP are the integro-differential equation or the system of such equations (finite or infinite); the partial-differential equation (PDE) or the system of such equations (finite or infinite), Kolmogorov equations; the initial conditions, and the integral boundary equation or a system of such equations (finite or infinite).

There exist several well-known methods to solve the BVP.

It is natural to think that partial differential equations (PDE) are principal parts of the BVP in the system (finite or infinite).

Unexpectedly, it turned out that in [1-3] and in the present paper, this part of the BVP is unnecessary.

Is it peculiar? Yes, it is.

That is why some of our colleagues doubt the existence of such an approach. Our alternative method shows that one can investigate all (or almost all) SM systems in QT and MTR without solving the PDE-s.

Hence, the most significant part (for the past 67 years after Cox published his original paper in 1955) of the above BVP and the PDE-s turned out to be redundant. It is surprising and intriguing scientifically, methodologically and philosophically. In fact, it is somewhat instructive.

We think that our approach shows the intrinsic probabilistic nature of the SM process, and it is impossible to discover these phenomena by studying the BVP without fully comprehending the considered systems in probabilistic terms.

The method is surprisingly simple and effective.

The starting point of our approach in the present paper, as well as in [1-3], is the examining of the considered system simultaneously in two-time epochs: 1) current time epoch t and 2) its previous time epoch $t-x,$ where x is one of the possible values of the very supplementary variable.

This consideration allows us to prove the theorem and all the other expressions.

As for the systems of partial differential equations, we decided to keep them for quadruple reasons: firstly, to show the complexities associated with the solution of such a system not using our novel method; secondly, for the reader, if one so desire, can check the consistency of new results with the previous work done other authors; thirdly, if someone desires to find unknown functions, using numerical methods, sometimes it can be done employing corresponding system of PDE-s, instead of finding inversion of Laplace transform of any expression obtained using analytical methods, including ours as well; fourthly, our method gives a general form to solution of various PDE-s, which often arise in the problems of modern science and technologies.

We believe that leading experts in the field will develop this approach further in various directions.

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ინფორმატიკა

გეივერის აღდგენადი პარალელური სისტემა რეზერვის დაყოვნებით

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გეივერის პარალელური სისტემა აღდგენით და რეზერვის დაყოვნებით გამოკვლეულია დამატებითი ცვლადის მეთოდის გამოყენებით, წმინდა ალბათურ არგუმენტირებასთან ერთად. ეს მიდგომა მნიშვნელოვნად ამარტივებს კლასიკური სისტემის საიმედოობის ანალიზს. ამავე დროს, ის იძლევა კერძოწარმოებულიანი დიფერენციალური განტოლებებისა და მათი სისტემების ზოგადი ამოხსნის მოხერხებულ ფორმას.

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