საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 18, №1, 2024
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 18, no. 1, 2024

Mathematics

Remarks on Invertible Binomial Singularities

Alexander Elashvili[**], Mamuka Jibladze[**], Giorgi Khimshiashvili[*]

[*] *School of Business, Technology and Education, Ilia State University, Tbilisi, Georgia*
[**] *A. Razmadze Mathematical Institute, Ivane Javakhishvili Tbilisi State University, Tbilisi, Georgia*

(Presented by Academy Member Tornike Kadeishvili)

Certain algebraic objects associated with an isolated quasihomogeneous plane curve singularity and their relations to other invariants of such a singularity are studied. Main attention is given to the so-called flex algebra and derivation Lie algebra of an isolated plane curve singularity. In particular, it is proved that the flex algebra is a finite dimensional commutative algebra and the Lie algebra of its derivations is solvable. For invertible binomial singularities, dimensions of their flex algebras are calculated and the structure of the latter is described in some detail. A plausible conjecture suggested by the obtained results is presented. *© 2024 Bull. Georg. Natl. Acad. Sci.*

isolated singularity, invertible polynomial, Milnor algebra, derivation Lie algebra, Hessian

1. Algebraic Invariants of Isolated Hypersurface Singularities

Let $\{f=0\}$ be an isolated plane curve singularity at the origin in $\mathbb{C}^2$ defined by a bivariate complex polynomial f. Denote by (J_f) the ideal generated by the partial derivatives of f in the commutative associative algebra $\mathbb{C}[\![2]\!]$ of complex formal power series in two indeterminates. In this situation, the factor-algebra $\mathbb{C}[\![2]\!]/(f,J_f)$ is finite-dimensional [1]. It is denoted by A_f and called the moduli algebra of f [1]. Its dimension is traditionally denoted by $\tau(f)$ and called the Tyurina number of f. The Lie algebra of derivations of A_f is usually denoted by L_f and its dimension is denoted by $\lambda(f)$. The latter two invariants introduced by S.S.-T. Yau in [2] (cf. a detailed review in [3]) have been actively studied in the last two decades by a number of researchers including the authors of this paper [4, 5]. In particular, it was shown in [4] that for the so-called simple singularities in the sense of Arnold [1], the Lie algebra L_f is a complete invariant of the analytic isomorphism class of a simple singularity f, excluding the pair A_6 ($\{x^7+y^2=0\}$) and D_5 ($\{x^2y+y^4=0\}$) [4]. Later on, similar results have been obtained for a wider class of the so-called *invertible binomial and trinomial singularities* (see, e.g., [3, 6]). In view of these results, finding other local derivation Lie algebras which would classify singularities in wider classes was recognized as a topical open problem (mentioned, e.g., in [6: 313, Question 4.1).

Motivated by this problem we study in some detail analogous concepts and results in the case of isolated singularities of bivariate binomials. Specifically, we consider another commutative associative algebra F_f and its derivation Lie algebra H_f, which are related to the so-called flexes of germ f as above. We establish certain of their properties which are relevant for our purposes (Theorem 1 and Proposition 2) and show that, in some cases, these algebras can be used to classify singularities considered. In particular, we prove that the Lie algebra H_f distinguishes the notorious pair A_6, D_5 (Proposition 4). We also compute the dimensions of F_f and H_f, and present more detailed information for the class of so-called *invertible binomial singularities* [7]. Along these lines we present an example showing that Lie algebra H_f is not a complete invariant of the analytic type for invertible binomial singularities. Moreover, computations presented in Section 4 enable us to verify one of our original conjectures (Theorem 2) and prove that the flex Lie algebra is solvable for the class of invertible binomial singularities (Theorem 3), which exhibits remarkable – known result of S.S.-T. Yau [2] and suggests a natural problem of characterizing those solvable Lie algebras which arise in this way.

2. Background and Main Results

We proceed with presenting the precise definitions and main results. For an isolated quasihomogeneous plane curve singularity $\{f=0\}$, we consider a planar map-germ $T_f=\left(f,h_f\right)$, where h_f is the Hessian of f, and call T_f the flex map of f. The local algebra of T_f at the origin is generically a finite-dimensional commutative associative algebra F_f. We call it the *flex algebra* of f since its dimension $\gamma\left(f\right)$ is equal to the number of complex flexes of the plane curve germ $\{f=0\}$ [6]. Finally, we denote by H_f the Lie algebra of derivations of F_f and call it the *flex Lie algebra* of f. Its structure and dimension $\sigma\left(f\right)$ are our main objects of interest. In particular, we provide some information about $\sigma\left(f\right)$ for a certain class of plane curve singularities introduced in the sequel.

In the case of real polynomial f, the flex algebra F_f describes some important features of the so-called *parabolic line* on the graph of f [6]. Another related invariant of f is the so-called *cusp-degree,* $\text{c-}\deg T_f$, of the flex map T_f at the origin, which is equal to the number of complex cusps in a generic stable deformation of T_f [8]. One of the aims of this note is to clarify relations between these invariants of the germ f by computing them for the so-called *invertible binomial singularities* studied in big detail in [7] (cf. also [6]). Such singularities are quasihomogeneous and their invariants can often be computed and/or estimated in terms of their quasihomogeneous types [9], which has appeared very helpful for our purposes.

Remark 1. It should be noted that several analogs of moduli algebras involving factorization over the ideals generated by certain minors of the Hessian matrix of f, and their derivation Lie algebras have been discussed in [3, 6, 10, 11]. However, to the best of our knowledge, the flex algebra F_f and the flex Lie algebra H_f introduced above, have not been considered neither in [3, 6, 10, 11] nor in other papers concerned with these topics. In the sequel we describe in some detail the structure of Lie algebra H_f for invertible binomial singularities.

3. Isolated Critical Points of Quasihomogeneous Polynomials

We proceed by presenting the definitions and certain properties of quasihomogeneous polynomials and invertible singularities which are reproduced from [1].

Definition 1 [1]. A polynomial f is called quasihomogeneous of qh type $(w_1, w_2, \ldots, w_n; d)$, for positive rational numbers $w_1, w_2, \ldots, w_n, d$, if for any complex number λ, the equality $f\left(\lambda^{w_1} x_1, ..., \lambda^{w_n} x_n\right) = \lambda^d f(x_1, \ldots, x_n)$ holds. The number w_k is called the weight of the variable x_k, $k = 1, \ldots, n$.

Obviously, any homogeneous polynomial P is quasihomogeneous with all weights equal to 1, and in this case $\boldsymbol{w}$-degree $\boldsymbol{w}$-deg P coincides with its usual algebraic degree deg P.

Example 1. The polynomial $f_1 = x^4 + xy^5$ is quasihomogeneous of type (5, 3; 20) and the germ $\{f_1 = 0\}$ has an isolated singularity at the origin of $\mathbb{C}^2$ which is not analytically isomorphic to any simple singularity.

Example 2. The polynomial $f_2 = x^4 + x^2 y^5$ is quasihomogeneous of type (5, 2; 20) but the germ of its singular set contains (the germ of) the complex line $\{x = 0\}$, so f_2 does not yield an isolated plane curve singularity. The germ f_2 provides an example of the so-called isolated line singularities introduced and studied by D. Siersma [1]. Some of our considerations and results can be extended to the latter class of singularities but we won't dwell upon that in this paper.

Definition 2 [1]. An endomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ is called quasihomogeneous of the type $(\boldsymbol{w}; \boldsymbol{d})$ if each of its components f_i is a quasihomogeneous polynomial of $\boldsymbol{w}$-degree d_i for one and the same system of positive rational weights $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$.

Clearly, the class of quasihomogeneous endomorphisms contains gradients of the quasihomogeneous polynomials. For example, gradf_1 is quasihomogeneous of the type (5, 3; 15, 17).

For a quasihomogeneous endomorphism F, its kernel is defined in the same way as in the linear case, i.e., $\ker F = \{x \in \mathbb{C}^n : F(x) = 0\}$. The kernel is invariant with respect to the nonlinear action of $\mathbb{C}^* = \mathbb{C} \setminus 0$, $x \mapsto \left(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n\right)$, where w_i are weights of the variables. The $\boldsymbol{w}$-orbit of a point $P \in \mathbb{C}^n$ is defined as the set $\mathbb{C}^* \cdot P$. If the kernel is nontrivial, it consists of $\boldsymbol{w}$-orbits, hence is non-compact and F is not a proper mapping. We are mostly interested in the cases where the kernel is trivial. For example, this holds for the endomorphisms gradf_1 and gradf_2. For $n = 2$, it is easy to compute the curvatures of $\boldsymbol{w}$-orbits in $\mathbb{R}^2$ and make a sketch of the corresponding $(\boldsymbol{w}; \boldsymbol{d})$-foliation. The following general proposition is used below for proving finite multiplicity of the flex map T_f of an invertible binomial singularity f.

Proposition 1 [1]. The kernel of a quasihomogeneous endomorphism F is trivial if and only if F is proper. If this is the case, then F has finite multiplicity at the origin.

We also present another general result on the multiplicity of the flex-map.

Proposition 2. If f is an isolated quasihomogeneous plane curve singularity of order not less than 3, then $\mu(f) \leqslant \gamma(f)$.

This result follows from the explicit formulae for these dimensions given in [1, 9]. This suggests that the Lie algebra H_f may contain more information than L_f and may be used to classify larger classes of singularities beyond the simple ones. In the next section we present more detailed results in the case of isolated binomial singularities.

4. Flex Algebras and Derivation Lie Algebras of Binomial Singularities

For our purposes it is sufficient to use the well-known fact that the class of invertible binomial singularities consists of the three series having the normal forms listed below. We compute our invariants for each of those normal forms. The computations are straightforward but tedious so we omit the details and present the relevant conclusions only. The following result, proven using case-by-case analysis and Proposition 1, serves as a necessary background for our considerations.

Theorem 1. For each invertible binomial singularity f, the flex algebra F_f and flex Lie algebra H_f are finite-dimensional.

Remark 2. For quasihomogeneous singularities in more than two variables this is not true since the ideal defining F_f has only two generators. In fact, it can also happen that F_f is of codimension less than 2, since there exist quasihomogeneous polynomials f with the hessian h_f proportional to its "parent function" f. For example, this is the case for polynomial $f = x^3 + y^3 + z^3 - 3xyz$ defining the singularity $\tilde{E}_6$. There also exist polynomials for which the hessian h_f is a nontrivial polynomial of the function f, as it happens for example for isoparametric two-dimensional surfaces. In all such cases the pair (f, h_f) is not a regular sequence in $\mathbb{C}[\![3]\!]$ and the flex-algebra F_f is infinite-dimensional. Comprehensive description of the germs with infinite-dimensional flex algebra is an interesting and practically unexplored task.

Besides being relevant by its own, Theorem 1 suggests several natural generalizations. We present below the most evident one concerned with invertible hypersurface singularities of arbitrary dimension discussed in [7]. Its formulation uses the notion and some properties of isolated singularity of complete intersection (ICIS) which can be found in [9]. In particular, for such an ICIS the so-called Milnor number is defined [9]. In the quasihomogeneous case the Milnor number can be effectively computed from the qh type of the mapping defining the ICIS considered. As is well known, each invertible binomial singularity is defined by a quasihomogeneous polynomial, hence its flex map is a quasihomogeneous mapping into the plane. Moreover, each such singularity has the so-called dual singularity belonging to the same class [7]. The following statement holds true in many examples, so we formulate it as a plausible conjecture.

Conjecture 1. For any invertible hypersurface singularity germ $\{f = 0\}$ in $\mathbb{C}^n$, its flex map $T_f = (f, h_f)$ defines an ICIS in $\mathbb{C}^n$.

If this is the case, the Milnor number of the arising ICIS can be effectively computed from the quasihomogeneous type of f [9]. According to Theorem 1, the dimension $\sigma(f)$ is finite for each invertible binomial singularity. To say more about it, and about the algebras H_f in general in what follows, we are going to present explicit monomial bases in the algebras F_f. In what follows, a, b are natural numbers both not less than 2.

Case 1 (Pham series) $f(x, y) = x^a + y^b$. Here weights are $\left(\frac{1}{a}, \frac{1}{b}\right)$, the Hessian is $h_f = x^{a-2} y^{b-2}$.

The highest nonzero powers of variables in F_f are respectively x^{2a-3} and y^{2b-3} since in this case $x^{2a-2} = -x^{a-2} y^b = -y^2 h_f$ and similarly $y^{2b-2} = 0$.

It follows that the monomial basis in F_f is given by the following set of monomials: $\left\{x^i y^j, x^{a+i} y^j = -x^i y^{b+j} \mid 0 \leqslant i \leqslant a-3, 0 \leqslant j \leqslant b-3\right\} \cup \left\{x^{a-2+i} y^j \mid i=0,1, 0 \leqslant j \leqslant b-3\right\} \cup \left\{x^i y^{b-2+j} \mid 0 \leqslant i \leqslant a-3, j=0,1\right\}$, hence $\gamma(f) = 2(a-2)(b-2) + 2(b-2) + 2(a-2) = 2(ab-a-b)$.

Case 2 (Loop series) $f(x,y) = x^a y + y^b x$. Here the weights are $\left(\frac{b-1}{ab-1}, \frac{a-1}{ab-1}\right)$, and the Hessian is $h_f = ab(ab-a-b-1)x^{a-1}y^{b-1} - a^2 x^{2(a-1)} - b^2 y^{2(b-1)}$.

Since $y^b x = -x^a y$ in F_f, we easily obtain $x^{2a-1}y = -x^a y^b = xy^{2b-1}$, while multiplying h_f by xy we obtain that $ab(ab-a-b-1)x^a y^b = a^2 x^{2a-1} y + b^2 y^{2b-1} x$ in F_f. It follows that we have $ab(ab-a-b-1)x^a y^b = -\left(a^2+b^2\right)x^a y^b$. But the quantity $ab(ab-a-b-1)$ is nonnegative for $(a-1)(b-1) \geqslant 2$, while $ab(ab-a-b-1) = -4$ and $-\left(a^2+b^2\right) = -8$ for $a=b=2$, so that in fact $x^{2a-1}y = x^a y^b = xy^{2b-1} = 0$ in F_f. Hence, $a^2 x^{3a-2} = ab(ab-a-b-1)x^{2a-1}y^{b-1} - b^2 x^a y^{2b-2} = 0$ and similarly $y^{3b-2} = 0$. All this gives a monomial basis in F_f as follows: the following set $\left\{x^i y^j, x^{a-1+i} y^j = -x^i y^{b-1+j} \mid 1 \leqslant i \leqslant a-1, 1 \leqslant j \leqslant b-1\right\} \cup \left\{x^i \mid 1 \leqslant i \leqslant 2(a-1)\right\} \cup \left\{y^j \mid 1 \leqslant j \leqslant 2(b-1)\right\} \cup \{1\}$ admits a single linear dependence, namely $a^2 x^{2(a-1)} + b^2 y^{2(b-1)} - ab\left((a-1)(b-1)-2\right)x^{a-1}y^{a-1} = 0$, as the left hand side here is equal to h_f. Hence $\gamma(f) = 2(a-1)(b-1) + 2(a-1) + 2(b-1) + 1 - 1 = 2(ab-1)$.

Case 3 (Chain series) $f(x,y) = x^a y + y^b$. Here weights are $\left(\frac{b-1}{ab}, \frac{1}{b}\right)$ and the Hessian is $h_f = ab(a-1)(b-1)x^{a-2}y^{b-1} - a^2 x^{2(a-1)}$.

Thus, in F_f one has $y^b = -x^a y$ and $ax^{2a-2} = b(a-1)(b-1)x^{a-2}y^{b-1}$. This then implies $ax^{a-2}y^b = -ax^{2a-2}y = -b(a-1)(b-1)x^{a-2}y^b = b(a-1)(b-1)x^{2a-2}y$. Since one has $a \neq -b(a-1)(b-1)$ for $a,b>1$, actually $x^{2a-2}y = x^{a-2}y^b = 0$ in F_f. Then also $y^{2b-1} = y^b y^{b-1} = -x^a y^b = 0$ and $ax^{3a-2} = ax^a x^{2a-2} = b(a-1)(b-1)x^{2a-2}y^{b-1} = 0$. We thus obtain that $\left\{x^i y^j, x^{a+i} y^j = -x^i y^{b-1+j} \mid 0 \leqslant i \leqslant a-3, 1 \leqslant j \leqslant b-1\right\} \cup \left\{x^{a-2+i} y^j \mid i=0,1, 1 \leqslant j \leqslant b-1\right\} \cup \left\{x^i \mid 0 \leqslant i \leqslant 2a-3\right\}$ is a monomial basis of F_f. This in particular implies the equality $\gamma(f) = 2(a-2)(b-1) + 2(b-1) + 2a - 2 = 2(a-1)b$.

The results of these computations enable us to obtain comprehensive information on the structure of flex Lie algebras and prove several remarkable conclusions. To give an idea of their proofs we describe in some detail the structure of those algebras for A_6 and D_5 singularities, which yields the following result which seems interesting in view of the open problem mentioned in Section 1.

Proposition 4. The flex Lie algebras of A_6 and D_5 are non-isomorphic.

Proof. Corresponding polynomials are $x^7 + y^2$ (for A_6) and $x^2 y + y^4$ (for D_5). The flex algebras are as follows: Hessian of $x^7 + y^2$ is a scalar multiple of x^5, so that relations in $F_{x^7+y^2}$ are $x^5 = y^2 = 0$ and a basis is given by x^i, $x^i y$, with $i = 0,1,2,3,4$. Thus, dimension of this algebra is 10. Hessian of $x^2 y + y^4$ is $24y^3 - 4x^2$, so that the relations for $F_{x^2y+y^4}$ are as follows: $x^2 = 6y^3$ and $x^4 = x^2 y = y^4 = 0$. There is thus a basis consisting of y^i, xy^i, $i = 0,1,2,3$. This algebra is 8-dimensional. As for derivation Lie algebras, $H_{x^7+y^2}$ has two linearly independent semisimple elements $x\partial_x$ and $y\partial_y$, while in $H_{x^2y+y^4}$ the only semisimple elements are scalar multiples of the Euler derivation $3x\partial_x + 2y\partial_y$. In fact, it is also easy to calculate directly that dimension of $H_{x^7+y^2}$ is 11 while dimension of $H_{x^2y+y^4}$ is 9.

Remark 3. Because of the equalities $\left(e^{\frac{\pi i}{3}}x+y\right)^3+\left(e^{\frac{2\pi i}{3}}x-y\right)^3=3\sqrt{3}i\left(x^2y+xy^2\right)$ and $\left(x+\sqrt{3}y\right)^3+\left(-x+\sqrt{3}y\right)^3=6\sqrt{3}\left(x^2y+y^3\right)$, the polynomials x^3+y^3, x^2y+xy^2, x^3+xy^2 define equivalent germs. Similarly, $(ix+iy)^4+\left(e^{\frac{\pi i}{4}}x-e^{\frac{\pi i}{4}}y\right)^4=8\left(x^3y+xy^3\right)$ shows that the germs of x^4+y^4 and x^3y+xy^3 are equivalent. It then follows that the flex algebras, and a fortiori flex Lie algebras in the corresponding cases are isomorphic. Further, we could not distinguish the algebras F_f for $f=x^n+y^n$, $x^{n-1}y+xy^{n-1}$ and $x^{n-1}y+y^n$, $n>4$. With these exceptions, in all cases that we have computed the algebras F_f and H_f, they come out pairwise non-isomorphic. In fact, in all these cases the algebras H_f have different lists of dimensions of lower central series of the nilradical. Let us also note that, although the germs of x^2y+xy^n and x^2y+y^{2n-1}, as well as x^ny+y^2 and x^2+y^{2n} are equivalent, they have non-isomorphic H_f algebras, a fortiori non-isomorphic F_f algebras.

As a by-product we were also able to prove the following result which verifies a conjecture formulated on the base of earlier sporadic computations.

Theorem 2. For each invertible binomial singularity, one has the inequality $\gamma(f)\leqslant\sigma(f)+1$.

Proof. Consider the subspace $F_fe:=\left\{ue \,\middle|\, u\in F_f\right\}$ of H_f, where $e=w_xx\partial_x+w_yy\partial_y$. This is a Lie subalgebra of dimension one less than the dimension of F_f. Indeed, for quasihomogeneous elements $u,v\in F_f$ one has $[ue,ve]=\left(\deg(v)-\deg(u)\right)uve$; moreover $ue=0$ implies $ux=uy=0$ since $e(x)=w_xx$ and $e(y)=w_yy$. It follows that the annihilator Z_e of e in F_f coincides with the annihilator of the maximal ideal of F_f. Calculations in Cases 1–3 above show that this annihilator is always one-dimensional: for x^a+y^b with $a,b>2$ it is spanned by the element $x^{a-3}y^{2b-3}=-x^{2a-3}y^{b-3}$ and for x^a+y^2 with $a>2$ by $x^{a-3}y$; for x^ay+xy^b by the element $\alpha x^{3a-3}=x^{2a-2}y^{b-1}=-x^{a-1}y^{2b-2}=\beta y^{3b-3}$ and for x^ay+y^b by $x^{a-3}y^{2b-2}=-x^{2a-3}y^{b-1}=\gamma x^{3a-3}$, where α, β, γ are certain scalars depending on a, b. Thus, $F_fe\cong F_e/Z_e$ has dimension $\gamma(f)-1$.

The following result exhibits interesting analogy with the well-known result of S.S.-T. Yau for derivation Lie algebras of moduli algebras [2] and suggests to try to characterize the solvable Lie algebras arising in this way.

Theorem 3. For any invertible binomial singularity f, the flex Lie algebra H_f is solvable.

Proof. Both the flex algebra F_f and the Lie algebra H_f can be graded by eigenvalues of the Euler derivation. We will prove that the negative degree part is zero, while degree zero part is a torus, i.e., consists of commuting semisimple elements.

Weight of a homogeneous derivation D is $w_{D(x)}-w_x=w_{D(y)}-w_y$. Without loss of generality, we may assume that weights of variables satisfy $w_x\leqslant w_y$. Then F_f possesses no homogeneous elements of positive weight less than w_x, hence $D(x)$ must be a scalar. On the other hand, if x^n is the largest nonzero power of x in F_f, we must have $(n+1)x^n D(x) = D\left(x^{n+1}\right) = 0$, so that $D(x) = 0$. Moreover $D(y)$ must be a scalar multiple λx^k of some x^k with $kw_x < w_y$. Note that if $D \neq 0$ then necessarily $k > 0$ since $(m+1)y^m D(y) = D\left(y^{m+1}\right) = 0$ for the largest nonzero power y^m of y in F_f.

It suffices to consider the cases of the Pham, loop and chain series above.

In the Pham case $x^a + y^b$ the weights are $w_x = \frac{1}{a}$, $w_y = \frac{1}{b}$. As $h_f = ab(a-1)(b-1)x^{a-2}y^{b-2}$, the derivation D above must satisfy the equalities $0 = D\left(x^{a-2}y^{b-2}\right) = (a-2)x^{a-3}y^{b-2}D(x) + (b-2)x^{a-2}y^{b-3}D(y) = \lambda(b-2)x^{a+k-2}y^{b-3}$. According to the calculations in 1) this implies $a+k-2 \geqslant 2a-2$, i.e., $k \geqslant a$. But then $kw_x < w_y$ cannot hold, since $kw_x \geqslant aw_x = 1$ while $w_y < 1$.

In the loop case $f = x^a y + xy^b$, we have $w_x = \frac{b-1}{ab-1}$, $w_y = \frac{a-1}{ab-1}$. Here we have $h_f = ab(ab-a-b-1)x^{a-1}y^{b-1} - a^2x^{2(a-1)} - b^2y^{2(b-1)}$, and $D\left(h_f\right) = 0$ with D as above translates into the equality $ab(b-1)(ab-a-b-1)x^{k+a-1}y^{b-2} = 2b^2(b-1)x^k y^{2b-3}$. Now calculations in 2) show that $x^{k+a-1}y^{b-2} = x^{k-1}x^a yy^{b-3} = -x^{k-1}xy^b y^{b-3} = -x^k y^{2b-3}$ for all $k > 0$ and $b > 2$, so that we must have $a(ab-a-b-1) = -2b$, or $b = \frac{a^2+a}{a^2-a+2}$. Then $b > 2$ implies $a^2 + a > 2\left(a^2 - a + 2\right)$, hence, $a^2 - 3a + 4 < 0$ which never happens. Whereas if $b = 2$ we obtain $a(a-3)x^{k+a-1} = 4x^k y$, and again computations in 2) show that in this case x^j is linearly independent from any other nonzero monomials in F_f for $j < 2a-2$, so that we must have $k + a - 1 \geqslant 2a-2$, i.e., $k \geqslant a-1$. But then $kw_x < w_y$ will imply $(a-1)(b-1) < a-1$, which would give $b < 2$, contradiction.

In the chain case we must consider two subcases, when $f = x^a y + y^b$ with $w_x = \frac{b-1}{ab}$, $w_y = \frac{1}{b}$, and $f = xy^a + x^b$ with $w_x = \frac{1}{b}$, $w_y = \frac{b-1}{ab}$, to capture all possibilities for $w_x \leqslant w_y$.

In the first subcase, we must have $k(b-1) < a$. Since $f = 0$ in F_f, we must have $D\left(x^a y\right) = -D\left(y^b\right)$, which means $x^{a+k} = -bx^k y^{b-1}$. Now calculations in Case 3 show that x^i is linearly independent of all other nonzero monomials for $i < 2a-2$, while $ax^i = b(a-1)(b-1)x^{i-a}y^{b-1}$ for $i \geqslant 2a-2$. Thus necessarily $a + k \geqslant 2a-2$ and then $ax^{a+k} = b(a-1)(b-1)x^k y^{b-1}$. If $x^{a+k} \neq 0$, this would imply $b(a-1)(b-1) = -ab$ which is impossible, so actually $x^{a+k} = 0$, which again by Case 3 can only happen if $a + k \geqslant 3a-2$, i.e., $k \geqslant 2a-2$. But this together with $k(b-1) < a$ would imply $(2a-2)(b-1) < a$, that is, $b < 1 + \frac{a}{2a-2}$, which is also impossible since $a, b > 1$.

In the second subcase we must have $ka < b-1$. Here $D(f) = 0$ means $D\left(xy^a\right) = -D\left(x^b\right)$ which is zero, so $x^{k+1}y^{a-1} = 0$ in F_f. Again, using Case 3 (this time with x and y interchanged) we see that this can only happen if $k+1 \geqslant b$. This would then imply $(b-1)a \leqslant ka < b-1$ which cannot happen.

This concludes the proof that there are no derivations of negative degree. Let us now turn to derivations of degree zero. Here we must separately consider the cases when $w_x < w_y$ and when $w_x = w_y$.

If $w_x < w_y$, then arguing as above we get that $D(x) = \lambda x$ for some scalar λ. Moreover either $D(y) = \mu y$ for some scalar μ or $D(y) = \mu x^k$ with $\mu \neq 0$, $k > 1$ and $kw_x = w_y$. The first of these is fine since all derivations of the form $\lambda x\partial_x + \mu y\partial_y$ are semisimple and commute with each other. For the second we again consider the Pham, loop and chain series separately.

If $f = x^a + y^b$, then we must have $a = kb$ with $k > 1$. Since $x^{a-2}y^{b-2} = 0$ in F_f, we obtain $0 = D\left(x^{a-2}y^{b-2}\right) = (a-2)x^{a-3}y^{b-2}D(x) + (b-2)x^{a-2}y^{b-3}D(y) = \lambda(a-2)x^{a-2}y^{b-2} + \mu(b-2)x^{a+k-2}y^{b-3}$ so that $x^{a+k-2}y^{b-3} = 0$. By Case 1 this implies $a + k - 2 \geqslant 2a - 2$, that is $k \geqslant a$. But then we must have $a = kb \geqslant ab$ which is impossible since $b > 1$.

If $f = x^a y + xy^b$ then we must have $a - 1 = k(b-1)$ with $k > 1$. From Case 2 we know that $x^{2a-1}y = 0$ in F_f, hence $0 = D\left(x^{2a-1}y\right) = \lambda(2a-1)x^{2a-1}y + \mu x^{2a-1+k}$. It follows that $x^{2a-1+k} = 0$ which again by Case 2 implies $2a - 1 + k \geqslant 3\text{a} - 2$, or $k \geqslant a - 1$. Since in this case we must have $a - 1 = k(b-1)$, we conclude $a - 1 \leqslant (a-1)(b-1)$, which is only possible for $b = 2$, $k = a - 1$. Thus, we are left with the case $f = x^a y + xy^2$, $D(x) = \lambda x$, $D(y) = \mu x^{a-1}$. Then $D(f) = 0$ implies $(\lambda a + 2\mu)x^a y + \mu x^{2a-1} + \lambda xy^2 = 0$ and $D(h_f) = 0$ implies $D\left(a^2x^{2a-2} - 2a(a-3)x^{a-1}y + 4y^2\right) = 0$, which reduces to $a\left(a(a-1)\lambda - (a-3)\mu\right)x^{2a-2} = \left(a(a-1)(a-3)\lambda - 4\mu\right)x^{a-1}y$. Again from Case 2, x^{2a-2} and $x^{a-1}y$ have linearly independent representatives in F_f, so that $a(a-1)\lambda - (a-3)\mu = 0$ and $a(a-1)(a-3)\lambda - 4\mu = 0.$ Since $\mu \neq 0$, this implies that $\det\begin{pmatrix} a(a-1) & 3-a \\ a(a-1)(a-3) & -4 \end{pmatrix} = 0$, so that $a(a-1)^2(a-5) = 0$. Thus, only $a = 5$ is possible. But then $D(f) = 0$ gives $(\lambda + \mu)x^5 y + \lambda xy^2 + \mu x^9 = 0.$ And as a particular case of Case 2, in F_f we have $x^5 y = -xy^2 = \dfrac{25}{24}x^9$, hence $\left(\dfrac{25}{24} + 1\right)\mu = 0$, which contradicts $\mu \neq 0$.

If $f = x^a y + y^b$ then $a = k(b-1)$ with $k > 1$. From Case 3, $x^{2a-2}y = 0$ so that $0 = D\left(x^{2a-2}y\right) = \lambda(2a-2)x^{2a-2}y + \mu x^{2a-2+k}$, hence $x^{2a-2+k} = 0$. Also, by Case 3 this implies $2a - 2 + k \geqslant 3a - 2$, or $k \geqslant a$. As in the previous case, this gives $a = k(b-1) \geqslant a(b-1)$, so that $b = 2$ and $k = a$. Thus, we are left with the case $f = x^a y + y^2$, $D(x) = \lambda x$, $D(y) = \mu x^a$. Then $D(f) = 0$ implies that $(a\lambda + 2\mu)x^a y + \mu x^{2a} = 0$ and $D(h_f) = 0$ that $\lambda(a-2)x^{a-2}y - (a\lambda - \mu)x^{2a-2} = 0$. Now by a particular case of Case 3, in F_f we have $x^{a-2}y = \dfrac{a}{2a-2}x^{2a-2}$, which gives equations $a^2\lambda + 2(2a-1)\mu = 0$ and $a^2\lambda + 2(a-1)\mu = 0$. This is impossible since $\mu \neq 0$.

Finally, let us address the case $w_x = w_y$. Then any derivation D of weight zero must satisfy $D(x) = \alpha x + \beta y$, $D(y) = \gamma x + \delta y$.

In the Pham case $f = x^a + y^a$, $D(f) = 0$ gives $a(\alpha - \delta)x^a + a\beta x^{a-1}y + a\gamma xy^{a-1} = 0$ so that $\beta = \gamma = 0$ and $\alpha = \delta$.

In the loop case $f = x^a y + xy^a$, and here $D(f) = 0$ implies the equality $(a-1)(\alpha - \delta)x^a y + a\beta x^{a-1}y^2 + a\gamma x^2y^{a-1} + \gamma x^{a+1} + \beta y^{a+1} = 0$, so also here $\alpha = \delta$ and $\beta = \gamma = 0$.

In the chain case $f = x^a y + y^{a+1}$ the requirement $D(f) = 0$ gives us $a(\alpha - \delta)x^a y + a\beta x^{a-1} y^2 + (a+1)\gamma x y^a + \gamma x^{a+1} = 0$, so again $\alpha = \delta$ and $\beta = \gamma = 0$.

In all three cases we obtain that D must be a scalar multiple of the Euler derivation.

Remark 4. Computations, performed with the GAP system [12], suggest the following values for dimensions of the flex Lie algebras:

$\dim H_{x^a+y^b} = 3(ab-a-b-1)$ for $a,b>2$; $\dim H_{x^a+y^2} = 3a-8$ for $a>2$

$\dim H_{x^a y+xy^b} = 3(ab-2)$ for $a,b>2$; $\dim H_{x^a y+xy^2} = 5a-3$ for $a>2$; $\dim H_{x^2 y+xy^2} = 6$

$\dim H_{x^a y+y^b} = 3(ab-b-1)$ for $a,b>2$; $\dim H_{x^a y+y^2} = 5a-7$ for $a>1$

The detailed proofs and further elaborations on the above results is a matter of future work.

The authors acknowledge financial support of the Shota Rustaveli Georgian National Scientific Foundation by the Grant STEM-22-604 "Deformations of associative and Lie algebras and applications in singularity theory and physics".

მათემატიკა

შებრუნებად ბინომიალურ განსაკუთრებულობათა შესახებ

ა. ელაშვილი, მ. ჯიბლაძე**, გ. ხიმშიაშვილი***

**ილიას სახელმწიფო უნივერსიტეტი, ბიზნესის, ტექნოლოგიისა და განათლების ფაკულტეტი, თბილისი, საქართველო*
***ივანე ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი, ა. რაზმაძის სახ. მათემატიკის ინსტიტუტი, თბილისი, საქართველო*

(წარმოდგენილია აკადემიის წევრის თ. ქადეიშვილის მიერ)

შესწავლილია იზოლირებულ კვაზიერთგვაროვან ბრტყელ ერთგანზომილებიან განსაკუთრებულობასთან დაკავშირებული ახალი ალგებრული ობიექტები, გამოკვლეულია მათი კავშირი აღნიშნული განსაკუთრებულობის სხვა ინვარიანტებთან. ძირითადი ყურადღება ეთმობა იზოლირებული ბრტყელი ერთგანზომილებიანი განსაკუთრებულობის ე. წ. ფლექს-ალ-

გებრასა და მისი გაწარმოებების ლის ალგებრას. კერძოდ, დამტკიცებულია, რომ ფლექს-ალგებრა არის სასრულგანზომილებიანი კომუტატური ალგებრა, ხოლო მისი გაწარმოებების ლის ალგებრა ამოხსნადია. შებრუნებადი ბინომიური განსაკუთრებულობებისათვის გამოთვლილია ფლექს-ალგებრების განზომილებები და დეტალურადაა აღწერილი მათი სტრუქტურა. გამოთქმულია საგულისხმო ჰიპოთეზა, რომელზეც მიღებული შედეგები მიუთითებს.

REFERENCES

1. Arnold V., Gusein-Zade S., Varchenko A. (1985) Osobennosti differentsiruemykh otobrazhenii (Singularities of differentiable maps). Nauka, M. (in Russian).
2. Yau S.S.-T. (1986) Solvable Lie algebras and generalized Cartan matrices arising from isolated singularities, *Math. J.,* **191**: 489–506.
3. Hussain N. (2018) Survey on derivation Lie algebras of isolated singularities. *Methods and Applications of Analysis,* **25**(4): 307–322.
4. Elashvili A., Khimshiashvili G. (2006) Lie algebras of simple singularities. *J. Lie Theory,* **16**(4): 621–649.
5. Jibladze M., Novikov D. (2007) Unimodularity of Poincaré polynomials of Lie algebras for semisimple singularities. *Moscow Math. J.*, **7**(3): 481–487.
6. Hussain N., Yau S.S.-T., Zuo H. (2020) On the new k-th Yau algebras of isolated hypersurface singularities. *Math. J.,* **294**(1-2): 331–358.
7. Ebeling W., Gusein-Zade S. (2011) Monodromy of dual invertible polynomials. *Moscow Math. J.*, **11**(3): 463–472.
8. Fukuda T., Ishikawa G. (1987) On the number of cusps of stable perturbations of a plane-to-plane singularity. *Tokyo J. Math.,* **10**(2): 375–384.
9. Greuel G.-M., Hamm H. (1978) Invarianten quasihomogener vollständiger Durchschnitte. *Invent. Math.,* **49**(1): 67–86.
10. Giorgadze G., Khimshiashvili G. (2023) On the index of the gradient of a real invertible polynomial. *Proc. Steklov Inst. Math.,* **321**(1): 84–96.
11. Ma G., Yau S. S-T., Zuo H. (2023) Nonexistence of negative weight derivations of the local k-th Hessian algebras associated to isolated singularities. *Pacific J. Math.,* **323**(1): 129–172.
12. The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.12.2; 2022. (https://www.gap-system.org).

Received December, 2023