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Mathematics

On Proper Quasihomogeneous Endomorphisms

Teimuraz Aliashvili

School of Business, Technology and Education, Ilia State University, Tbilisi, Georgia

(Presented by Academy Member Elizbar Nadaraya)

We give an effective criterion of properness for quasihomogeneous endomorphisms of Euclidean space. We also show that in certain cases surjectivity of a proper quasihomogeneous endomorphism can be derived from the algebraic properties of its quasihomogeneous type. The main results are illustrated by a number of examples. *© 2024 Bull. Georg. Natl. Acad. Sci.*

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Description of the sets of solutions to systems of polynomial equations is one of the most fundamental algebraic problems. Many aspects of this problem can be reformulated in terms of geometric properties of associated polynomial mappings [1,2]. In particular, some important properties of polynomial systems, such as solvability, have natural reformulations in the language of polynomial mappings. In particular, solvability of polynomial system for arbitrary right-hand sides (this property is often called the *unconditional solvability* of the system) is equivalent to the surjectivity of the polynomial mapping defined by the left-hand sides of the given equations. Analogously, compactness of the sets of solutions for arbitrary right-hand sides is equivalent to properness of the corresponding polynomial mapping. The existence of continuous branches of solutions can be expressed in terms of the existence of a continuous *right inverse* of the corresponding polynomial map.

Thus, there arises a possibility to establish various properties of solutions to polynomial systems using results on the geometry of polynomial mappings. Since unconditional solvability is a nice property which is important in many applications, it is desirable to have effective methods of verifying the surjectivity of a given proper map. In particular, the famous Keller's Jacobian conjecture requires proving that an endomorphism with the constant Jacobian is surjective [3]. This conjecture remains unproven, which shows that proving surjectivity is a difficult and topical problem.

For homogeneous endomorphisms, it is known that surjectivity is closely related to properness, in particular, under some assumption properness implies surjectivity (see, e.g., [1,4]). The main aim of this paper is to obtain analogous results for real quasihomogeneous endomorphisms. To this end we first give an effective criterion of properness of real quasihomogeneous endomorphism (Theorem 1) and then describe some cases where surjectivity can be derived from the algebraic properties of the quasihomogeneous type (Theorem 2). The main results are illustrated by several typical examples. An important class of quasihomogeneous endomorphisms consists of the gradients of quasihomogeneous polynomials [2]. For this class, we obtain some further sufficient conditions of the surjectivity (Proposition 3 and Proposition 4).

We proceed by recalling some definitions and general results of nonlinear analysis. To place our discussion in an appropriate perspective we begin by discussing the case of homogeneous endomorphism. So let $F : R^n \to R^n$ be a homogeneous polynomial endomorphism of degree $p \geq 1$, i.e., $F(\lambda x) = \lambda^p F(x)$ for any $\lambda \in R$. For such maps one defines the *kernel* as

$$ker\, F = \left\{ x \in R^n : F\left(x\right) = 0 \right\}.$$

Note that $\ker F$ is a subset of R^n invariant with respect to the linear action of $R^* = R \setminus 0, \quad x \mapsto \lambda x$ by homotheties. Recall that for a homogeneous linear map $L : R^n \to R^n$, the theorem about rank states that L is injective if and only if it is surjective, i.e. $\ker L = 0 \Leftrightarrow Im\, L = R^n$. In that case, the map L is invertible and the inverse map is bounded.

This property is important in nonlinear analysis, when one solves nonlinear equations using linear approximations (e.g., the Newton's method) [1]. Moreover, for linear maps, both these properties are equivalent to properness. Indeed, non-injectivity in this case is equivalent to the fact that $\ker F$ contains a nontrivial linear subspace which is obviously non-compact, hence properness fails. Also, in the linear case each surjective map has a bounded right inverse which in this case is the usual (bilateral) inverse. Thus, in linear case properness, surjectivity and triviality of the kernel are all equivalent and we wonder if there are some relations between these properties in nonlinear case.

In particular, one is often interested in existence of the right inverse $H = \left\{ F^{-1} \right\}$ to an endomorphism F, i.e. such a map H that $F \circ H = id$. The map H exists if F is surjective, but it can be not unique and not continuous. In fact, it is composed of branches of an algebraic multi-valued map so the geometric picture in nonlinear case can be rather complicated.

From the view of applications (e.g., in nonlinear analysis) not only the existence of the right inverse $H = \left\{ F^{-1} \right\}$ to F is important. Maybe even more important is its boundedness, i.e. we ask whether one can choose H in such a way that $\left| H\left(z\right) \right| < C$ if $\left| z \right| \leq 1$ for some constant C. Such property is called the *Banach property* of F. The book [4] contains an example of homogeneous polynomial endomorphism of the plane for which $\left\{ F^{-1} \right\}_r$ exists (i. e. F is surjective), but it cannot be chosen as bounded. The following classical result clarifies the picture in the case of homogeneous endomorphism.

Proposition 1. (see, e.g., [1]) Let F be as above. Then F is proper if and only if $\ker F = \left\{ 0 \right\}$. Moreover, if F is proper and surjective, then any right inverse to F is bounded.

Hence, surjectivity of F guarantees existence and boundedness of right inverses to F. Thus, it is important to obtain sufficient conditions of surjectivity. Our first main result, Theorem 1 given below, generalizes Proposition 1 to a wider class of the so-called quasihomogeneous endomorphisms, for which we also give an effective sufficient condition of surjectivity in Theorem 2.

We proceed by reproducing the definitions and some properties of quasihomogeneous polynomials and quasihomogeneous endomorphisms which can be found, e.g., in [2,5,6].

Definition 1. Polynomial *f* is quasihomogeneous of the type $(w_1, w_2, ..., w_n; d)$ if there exist $(w_1, w_2, ..., w_n; d)$ rational numbers such that for any $\lambda \in R$ we have $f\left(\left(\lambda x_1^{w_1}\right), ..., \left(\lambda x_n^{w_n}\right)\right) = \lambda^d f(x_1, ..., x_n)$. The index w_k is called the weight of the variable x_k .

Example 1. Polynomial $f_1 = x^4 + xy^5$ is quasihomogeneous of the type (5, 3; 20).

Obviously, any homogeneous polynomial P is quasihomogeneous with all weights equal to 1, and in this case w-degree w - $deg\, P$ coincides with the usual degree $deg\, P$.

Definition 2. Endomorphism F is quasihomogeneous of the type $(w;d)$ if each of its components f_j is a quasihomogeneous polynomial of w - $deg\,ree\, d_i$ for one and the same system of positive rational weights $w = (w_1, w_2, ..., w_n)$.

It is easy to see that the class of quasihomogeneous endomorphisms contains the gradients of quasihomogeneous polynomials. For example, gradient grad f_1 is a quasihomogeneous mapping of the type (5, 3; 15, 17). The following example shows that the class of quasihomogeneous endomorphisms is wider than the class of quasihomogeneous gradients.

Example 2. The mapping $F_1 = \left(x^4 + xy^5, x^3y^3 - y^8\right): R^2 \to R^2$ is quasihomogeneous of the type (5, 3; 20, 24) but it cannot be represented as the gradient of any polynomial because it does not satisfy the condition of equality of the mixed derivatives of the second order. Indeed, $\frac{\partial}{\partial y}\left(x^4 + xy^5\right) = 5xy^4$ does not coincide with $\frac{\partial}{\partial x}\left(x^3y^3 - y^8\right) = -3x^2y^3$.

For a quasihomogeneous endomorphism F, its kernel is defined in the same way as in the homogeneous case, i. e. $ker\, F = \left\{x \in R^n : F(x) = 0\right\}$.

In quasihomogeneous case, the kernel is invariant with respect to nonlinear action of $R^* = R \setminus 0,\, x \mapsto \left(\lambda^{w_1}, x_1, ..., \lambda^{w_n} x_n\right)$, where w_i are the weights of variables. The quasihomogeneous w-orbit of a point P is defined as the set $R^* \cdot P$. In this case the orbits are curves in R^n with the natural parametrization given by the above action. Each orbit consists of two non-compact simple arcs (w-rays) which give a foliation of R^n with a singular point at the origin, which we call the $(w;d)$- foliation. For $n = 2$ and $n = 3$, it is easy to compute the curvatures of w- orbits and make a sketch of the corresponding $(w;d)$- foliation of the ambient space. If the kernel is nontrivial it consists of w- orbits and is non-compact. We are interested in the cases where the kernel is trivial, which is the case for the endomorphisms grad f_1 and F_1 .

Example 3. For $grad\, f_1$, it is easy to verify that the $(w;d)$- foliation of R^2 consists of coordinate axes, The kernel of $grad\, f_1$ is trivial and the curvature of w- orbit of a point (a,b) is equal to

$$k = \frac{w_1 w_2 (w_2 - w_1) ab}{\left(w_1^2 a^2 + w_2^2 b^2\right)^{\frac{3}{2}}} = \frac{30ab}{\left(25a^2 + 9b^2\right)^{\frac{3}{2}}}$$

It follows that $|grad\ f_1|$ tends to infinity as the point P tends to infinity along any of w -orbits, which by Hadamard's theorem guarantees properness and surjectivity of $grad\ f_1$.

Example 4. It is also easy to see that endomorphism $F_1 = \left(x^4 + xy^5, x^3y^3 - y^8\right)$ has trivial kernel and one can verify that F_1 is proper.

Example 5. Endomorphism $F_2 = \left(x^4 + xy^5, x^3y^3 + y^8\right)$ has nontrivial kernel which coincides with the curve $\left\{y^5 = -x^3\right\}$ so F_2 is not proper.

Definition 3. For any continuous map $F : X \to Z$, one defines the non-properness set $F : X \to Z$ as consisting of those points z_0 which have arbitrary small neighborhoods with compact closures whose inverse images are unbounded. In other words, $z_0 \in S(F)$ iff there exists a sequence $z_n \to z_0$ admitting preimages $x_n = F^{-1}(z_n)$ which form an unbounded sequence, $|x_n| \to \infty$. Obviously, a map F is proper if and only if $S(F) = \varnothing$.

We give now generalization of Proposition 1 to the quasihomogeneous case.

Theorem 1. Let F be a quasihomogeneous endomorphism. Then F is proper if and only if $ker\, F = \{0\}$. Moreover, if F is proper and surjective, then any right inverse to F is bounded.

This result is effective since the triviality of kernel can be algorithmically verified. Indeed to show the triviality of kernel it is sufficient to show that a certain system of polynomial equations has no real solutions. As is explained in [7] this can be done using the so-called signature formula for the cardinality of finite algebraic set given in [8].

The following simple lemma will be used in the proof of Theorem 1.

Lemma 1. For a system of positive weights w as above and any point P different from the origin, the w-orbit P intersects the unit sphere S in R^n at a single point P_w.

Indeed, there exists the unique number $r(P)$ such that $r(P)P$ belongs to the unit sphere S, which is found from the condition $r^d |P| = 1$.

Proof of Theorem 1. If $ker\, F \neq 0$ and $x_o \in ker\, F \setminus \{0\}$, then the whole unbounded w- orbit Rx_0 is sent to the origin. So the preimage of the origin cannot be compact. Assume now that $ker\, F = 0$. Let $S = \left\{x \in X : |x| = 1\right\}$ be the unit sphere as above.

No point of S is sent to the origin **0**, so $0 \notin F(S)$. Of course, $F(S)$ is compact. Hence, the point origin is separated from $F(S)$, i. e. $inf_{z \in F(S)} |z| = A > 0$.

By Lemma 1 any $x \in X\{0\}$ can be written in the form $x = r(x)x_0$, where $x_0 \in S$. We have then $F(x) = r^d F(x_o)$ and hence $|F(x)| \geq r^d A = A \cdot |x|^d$. We see that $|F(x)| \to \infty$ as $|x| \to \infty$. By Hadamard's criterion [1], this is equivalent to the properness of F.

Assume now that F is proper and surjective and suppose that some its right inverse H is unbounded. Then there should exist a sequence $z_n \in Z$ with norms bounded by 1 such that $x_n = H(z_n) \to \infty$ (i. e. is unbounded). Eventually passing to a subsequence, we can assume that the sequence $\{z_n\}$ is convergent to same z_0. But this would imply that $z_o \in S(F)$, in contradiction to the assumption about properness of F. The proof of Theorem 1 is complete.

In this section we discuss the problem of surjectivity of a proper quasihomogeneous endomorphism. Obviously, one cannot expect equivalence of the properties $ker\, F = 0$ and $Im\, F = R^n$. Indeed, the one-dimensional maps $F(x) = x^{2k}$ have trivial kernels but $Im\, F = \{z \geq 0\}$. However, the implication

$$ker\, F = 0 \Rightarrow Im\, F = R^n \tag{1}$$

holds for certain quadratic homogeneous endomorphisms $F : R^n \to R^n$. Namely, A. Izmailov and A. Tret'yakov have proved that for homogeneous quadratic endomorphisms and $n = 1, 2, 3,$ the implication (1) holds true [4]. At the same time, the following result from [9] shows that this is wrong in higher dimensions.

Proposition 2. Assume that either $p \geq 3,\ n \geq 2$ or $p = 2, n \geq 4$. Then there exists a polynomial homogeneous map $R^n \to R^n$ of degree P which is surjective and not proper.

However, it remains unclear if such examples exist in the class of gradient mappings. We complement results of [4] by giving a sufficient condition for properness of gradient in quasihomogeneous case for $n = 2$.

Proposition 3. If a real quasihomogeneous polynomial f is nonnegative or nonpositive, then its gradient is proper and surjective.

It is well known that verifying the surjectivity of a proper endomorphism is usually difficult and there are only a few general results for homogeneous endomorphisms. Some of such results using the topological mapping degree of endomorphism were given in [7]. In particular using properties of mapping degree it is possible to prove that proper homogeneous endomorphism of odd degree is surjective [7]. Our next result gives a sufficient condition of surjectivity in the quasihomogeneous case, which generalizes the result mentioned in the previous sentence. It uses the concept of multiplicity of proper quasihomogeneous endomorphism and its relation to the mapping degree. All necessary definitions and auxiliary results on the multiplicity and mapping degree can be found in [7].

Theorem 2. If the multiplicity of proper quasihomogeneous endomorphism F is odd, then F is surjective.
Proof of Theorem 2. From the signature formula for the mapping degree proven in [8] follows that the absolute value of the local degree at the origin $deg_0\, F$ differs from the multiplicity by an even number so in this case the local degree of F is also odd. By Theorem 1 properness is equivalent to the triviality of kernel so endomorphism F restricted on big spheres avoids the origin, which implies that its global mapping degree $Deg\, F$ is well defined [7]. Lemma 1 implies that, in this case, one has $deg_0\, F = DegF$ so $deg\, F$ is odd, which by a well-known property of mapping degree implies that F is surjective and completes the proof.

This result is effective since there exists an explicit formula for the muliplicity of quasihomogeneous endomorphism in terms of its quasihomogeneous type given in Section 12 of the monograph [2]. We illustrate the use of this formula by the following example.

Example 6. Let us prove that the gradient mapping of polynomial $f_1 = x^4 + xy^5$ is surjective. As was mentioned, this polynomial is quasihomogeneous of the type (5, 3; 20). It is easy to verify that its gradient is a quasihomogeneous endomorphism of quasihomogeneous type (5, 3; 15, 17). The explicit formula from [2], mentioned above gives that its multiplicity is equal to $\frac{(20-5)(20-3)}{15}=17,$ so it is odd. We conclude that the gradient $grad\, f_1$ is surjective. In this case it is easy to verify surjectivity of the gradient. In fact, this gradient is an invertible endomorphism, i.e. it has a bilateral inverse.

Using Theorem 2 one can construct many examples of surjective proper endomorphisms in arbitrary dimension. In particular, a wide class of examples arises from the gradients of so-called invertible polynomials studied in papers on mathematical physics [7]. As an illustration we give one example in three dimensions, where we use both of our theorems.

Example 7. Consider trivariate polynomial $P(x,y,z)=x^3y+y^5z+z^7x$. It is easy to verify that this polynomial is of the quasihomogeneous type (29, 19, 11; 106). It follows that its gradient $grad\, P$ is of quasihomogeneous type (29, 19, 11; 77, 87, 95). Using Theorem 1 it is easy to verify that $grad\, P$ is proper. By the already mentioned formula, its multiplicity is $\mu=3\cdot5\cdot7=105$. Hence, Theorem 2 implies that $grad\, P$ is surjective. In this case direct verification of surjectivity requires proving unconditional solvability of a (3×3)-system of polynomial equations, which would be rather difficult.

Proposition 4. If a quasihomogeneous f polynomial has odd multiplicity then, for any monomial g of w- degree bigger than w - $deg\, f$, gradient of $f+g$ is surjective.

Indeed, as is proven in [2]. In this case $f+g$ can be transformed into f by a change of coordinates, so the result follows from our Theorem 2.

მათემატიკა

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თ. ალიაშვილი

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