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Informatics

Alternative Transient Solution for M/G/1/N Systems

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Semi-Markov models are fundamental components of classical queuing theory and the mathematical theory of reliability, serving as valuable tools for optimizing the efficiency of technical systems. This work investigates the complex interplay between reliability, capacities, inputs and outputs within various components of such systems. Specifically, the study focuses on the M/G/1/N system, which represents a mathematical model for a wide range of technical systems. The study employs diverse analytical methods, including supplementary variable and Laplace transform techniques. In this paper, a novel approach is used to study the M/G/1/N system. The novel probability method involves simultaneous consideration of the system at two distinct time moments: one coinciding with the time of observation of the system in repair and the other representing beginning of the mentioned repair. This approach yields solutions to the problem, offering new insights into the system's behavior. The research is focused on a machine repairman problem with a single server and a set of working elements, characterized by the element failure and repair times with general failure. Probability characteristics of the system are analyzed, leading to a system of partial differential equations with nonlocal boundary conditions, presenting a challenging boundary value problem in mathematical physics. An innovative solution method that eliminates the need for the complex partial differential equations has been presented.

supplementary variable method, semi-Markov model, purely probabilistic reasoning

Semi-Markov models constitute an important part of classical queuing theory and mathematical theory of reliability. The investigation of the relationships between reliabilities, capacities, inputs, and outputs within various components is of great interest to design engineers and managers, as it can significantly aid in optimizing the efficiency of technical systems. Since the finite-source model has applications in widely different situations, our research investigates the M/G/1/N system, as it represents the mathematical model of many important technical systems. To investigate these systems we have employed several analytical methods, such as the supplementary variable method and Laplace transform [1-7]. We have adopted a new method to study such a system. The method is based on simultaneous examination of the system at two different time points: one coinciding with the time of observation of the system in repair and the other representing the beginning of the mentioned repair [8-10].

To investigate the M/G/1/N system, we will examine a machine repairman problem with a single server and a set of working elements. The failure of each element takes place with intensity λ. Additionally, their repair (service) times are independent and identically distributed at random variables, with cumulative distribution function $G(x)$ and probability density function $g(x)$.

The state of the system at time t is described using following random variables:

$\widetilde{N}(t)$ denotes number of failed elements at time t;

$\xi(t)$ is elapsed repair time since the beginning of the repair job observed at time t.

We consider functions describing following probabilities:

$$P(t) = \mathbb{P}\left\{\widetilde{N}(t) = 0\right\}$$

$$p(n,x,t) = \lim_{h \to 0} \frac{1}{h} \mathbb{P}\left\{\widetilde{N}(t) = n, x \le \xi(t) < x + h\right\}.$$

As is already known from various sources [1-5], we can derive the subsequent integro-differential equation (1) and system of partial differential equations (2-3) by utilizing these functions.

$$\frac{dP(t)}{dt} = -N\lambda P(t) + \int_0^t p(1,x,t)\,dx, \tag{1}$$

$$\frac{\partial p(1,x,t)}{\partial t} + \frac{\partial p(1,x,t)}{\partial x} = -\left((N-1)\lambda + \eta(x)\right) p(1,x,t). \tag{2}$$

If $1 < n \le N$, then

$$\frac{\partial p(n,x,t)}{\partial t} + \frac{\partial p(n,x,t)}{\partial x} = -\left((N-n)\lambda + \eta(x)\right) p(n,x,t) + (N-n+1)\lambda p(n-1,x,t). \tag{3}$$

Boundary conditions are:

$$p(1,0,t) = N\lambda P(t) + \int_0^t p(2,x,t)\eta(x)\,dx \tag{4}$$

$$p(n,0,t) = \int_0^t p(n+1,x,t)\eta(x)\,dx \tag{5}$$

$$p(N,0,t) = 0, \tag{6}$$

where $\eta(x) = g(x)/\left(1-G(x)\right)$ is a service rate function.

As an initial condition, we assume that $P(0) = 1$.

The system (1-3) and (4-6) boundary conditions form a non-classical boundary value problem of mathematical physics with nonlocal boundary conditions.

Up to this point, a closed-form analytical solution for this problem has remained elusive. While one solution exists in terms of operational calculus, here we present a novel approach for obtaining transient solutions for the M/G/1/N system introduced above.

The main advantage of our approach lies in the following fact: we can derive an expression for $p(n,x,t)$ using purely probabilistic reasoning without consideration of the (2-3) system of partial differential equations. Our method begins with examining the system at two-time moments: one coinciding with the time of observation of the system in repair, t and the other representing beginning of the said repair $t-x$, with x representing one of the possible values of a supplementary variable. The analysis of the system over the time interval $(t-x, t)$ leads to the following theorem.

Theorem 1. If $1 \le n$ then the expression for the function $p(n,t,x)$ has the following form

$$p(n,x,t) = (1-G(x))\sum_{k=1}^{n} p(k,0,t-x)\,C_{N-k}^{n-k}\left(1-e^{-\lambda x}\right)^{n-k}\left(e^{-\lambda x}\right)^{N-n}. \tag{7}$$

The proof follows the same steps as our previous works [8-10].

Using Newton's binomial formula for $P\left(B_{n-k}(x,t)\right)$ we get

$$P\left(B_{n-k}(x,t)\right) = C_{N-k}^{n-k}\left(1-e^{-\lambda x}\right)^{n-k}\left(e^{-\lambda x}\right)^{N-n} = \sum_{i=0}^{N-k}(-1)^i C_{N-k}^{n-k} e^{-\lambda(N-n)x}\cdot C_{n-k}^{i} e^{-\lambda i x} =$$
$$= \sum_{i=0}^{n-k}(-1)^i C_{N-k}\left(N-n, n-k-i, i\right) e^{-\lambda(N-n+i)x},$$

where

$$C_{N-k}\left(N-n, n-k-i, i\right) = \frac{(N-k)!}{(N-n)!(m-k-i)!\,i!}.$$

And finally, from (7) with respect to the functions $p(n,x,t)$ we get the following relations

$$p(n,x,t) = (1-G(x))\sum_{k=1}^{n} p(k,0,t-x)\sum_{i=0}^{n-k}(-1)^i C_{N-k}\left(N-n, n-k-i, i\right) e^{-\lambda(N-n+i)x}. \tag{8}$$

At the next stage, we will use the expressions for $p(1,x,t)$ and $p(2,x,t)$ from (8) in equation (1) and in (4-5) boundary conditions.

After some transformations we have

$$\frac{dP(t)}{dt} = -N\lambda P(t) + \int_0^t p(1,0,t-x)e^{-\lambda(N-1)x}g(x)\,dx \tag{9}$$

$$\begin{aligned} p(1,0,t) &= N\lambda P(t) + (N-1)\int_0^t p(1,0,t-x)e^{-\lambda(N-2)x}g(x)\,dx - \\ &-(N-1)\int_0^t p(1,0,t-x)e^{-\lambda(N-1)x}g(x)\,dx + \int_0^t p(2,0,t-x)e^{-\lambda(N-2)x}g(x)\,dx \end{aligned} \tag{10}$$

$$p(n,0,t) = \sum_{k=1}^{n+1}\sum_{i=0}^{n+1-k}\int_0^t (-1)^i p(k,0,t-x)\,C_{N-k}\left(N-n-1, n+1-k-i, i\right)e^{-\lambda(N-n-1+i)x}g(x)\,dx. \tag{11}$$

Applying Laplace transform to (6), (9)-(11) and taking into account initial condition $P(0)=1$, we obtain linear algebraic equations with respect to values of $\overline{P}(s)$ and $\overline{p}(i,0,s)$, that are Laplace transforms of functions $P(t)$ and $p(i,0,t)$.

$$\begin{gathered} (s+N\lambda)P(s) - p(1,0,s)\,g\left(s+\lambda(n-1)\right) = 1 \\ N\lambda P(s) + p(1,0,s)\left[(N-1)g\left(s+\lambda(N-2)\right) - (N-1)g\left(s+\lambda(N-1)\right) - 1\right] + \\ +p(2,0,s)g(s+\lambda(N-2)) = 0 \\ p(N,0,s) = 0. \end{gathered} \tag{12}$$

For $1 < n < N$:

$$\sum_{k=1}^{m-1}\sum_{i=0}^{m+1-k}(-1)^i C_{n-k}(n-m-1,m+1-k-i,i)\,p(k,0,s)\,g\big(s+\lambda(n-m-1+i)\big)+$$
$$+p(m,0,s)\Big((n-m)\,g\big(s+\lambda(n-m-1)\big)-(n-m)\,g\big(s+\lambda(n-m)\big)-1\Big)+$$
$$+p(m+1,0,s)\,g\big(s+\lambda(n-m-1)\big)=0$$

Considering the expressions $\overline{P}(s)$ and $\overline{p}(i,0,s)$ from (12) we finally obtain

$$p(n,x,s)=\big(1-G(x)\big)\sum_{k=1}^{n}p(k,0,s)\sum_{i=0}^{n-k}(-1)^i C_{N-k}(N-n,n-k-i,i)\,e^{-\lambda(N-n+i)x}. \quad (13)$$

Conclusion

The above-mentioned discussion may be summed up by saying that in the stochastic models of reliability systems, there is often “hidden” a simple, pure probabilistic chance for their investigation. The basic point of the approach is the consideration of the constructed semi-Markov processes simultaneously at two-time instants: 1) the current time instant t, and coinciding with repair, 2) the previous time instant t-x, representing the beginning of the repair process, where x is one of the possible values of a supplementary variable. They significantly simplify the reliability analysis of the considered technical systems.

We believe that with a certain appropriate modification of the method, it will be possible to investigate other semi-Markov systems analogously.

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ინფორმატიკა

M/G/1/N სისტემის ალტერნატიული გადაწყვეტა გარდამავალ რეჟიმში

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ნახევრად მარკოვის მოდელები კლასიკური მასობრივი მომსახურების თეორიისა და საიმედოობის მათემატიკური თეორიის ფუნდამენტური კომპონენტებია, რომლებიც ტექნიკური სისტემების წარმადობის ოპტიმიზაციის ძლიერ ინსტრუმენტებს წარმოადგენს. ნაშრომის მიზანია სისტემის საიმედოობის, ეფექტიანობისა და შემავალ და გამომავალ ნაკადებს შორის კომპლექსური ურთიერთკავშირის შესწავლა. კონკრეტულად, კვლევაში ყურადღება გამახვილებულია M/G/1/N სისტემაზე, რომელიც წარმოადგენს მათემატიკურ მოდელს და გამოიყენება მრავალი პრაქტიკული ტექნიკური სისტემის აღსაწერად; გამოყენებულია სხვადასხვა ანალიზური მეთოდი, მათ შორის, დამატებითი ცვლადის მეთოდი და ლაპლასის ტრანსფორმაცია, ასევე, ახალი მიდგომა M/G/1/N სისტემის შესწავლისადმი. ახალი ალბათური მეთოდი გულისხმობს სისტემის ერთდროულად განხილვას ორ განსხვავებულ დროში: ერთი ემთხვევა სისტემაში მიმდინარე დროის მომენტს და მეორე წარმოადგენს მიმდინარე აღდგენის დასაწყისს. ეს მიდგომა იძლევა პრობლემის გადაწყვეტის შესაძლებლობას და მკითხველს სთავაზობს ახალ შეხედულებებს ასეთი სისტემის ქცევაზე. გაანალიზებულია სისტემის ალბათური მახასიათებლები, რასაც მივყავართ კერძოწარმოებულიან დიფერენციალურ განტოლებათა სისტემამდე არალოკალური სასაზღვრო პირობებით. ის წარმოადგენს მათემატიკური ფიზიკის რთულ სასაზღვრო ამოცანას. კვლევაში წარმოდგენილია ამოხსნის ახალი მეთოდი, რომელიც გამორიცხავს კერძოწარმოებულიანი დიფერენციალური განტოლებების ამოხსნის საჭიროებას.

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