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Mechanics

On the Extreme Properties of Moments of Inertia of Material Bodies

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A problem of homogeneous material of mass *m* with σ density is considered. The question arises as to what type of plane shape this material should take and how it should be located on the Oxy plane so that its moment of inertia relative to the Oz axis reaches extreme values. The problem solution suggests that the moment of inertia between convex plane figures relative to the Oz axis reaches its minimum value when it has the shape of a circle with the center at the point O and radius *R*. If a plane figure has the form of an ellipse, the moment of inertia relative to the Oz axis can reach a very large value. The same problem is considered when calculating the moment of inertia is calculated relative to any Oℓ axis passing through the center of masses, making an angle α ($\alpha \neq 0,\ \alpha \neq \pi$) with the Oz axis. The moment of inertia reaches its minimum value when the plane figure has the shape of an ellipse α and b is with semi-axes. In the three-dimensional case, the polar moment of inertia of the material of mass m and density σ reaches a minimum relative to pole O when the material has the shape of a core with center O and radius *R*.* *© 2024 Bull. Georg. Natl. Acad. Sci.*

material body, moment of inertia, maximum, minimum, circle, ellipse, core, Holder's inequality

The analysis of the motion of a material system shows that the relationship between the kinetic and kinematic elements of the bodies of the system is determined not only by the body masses included in the system, but also by their location in space. Therefore, in dynamics, special geometric characteristics of mass distribution, namely moments of inertia, are used to describe the rotational motion of bodies or mechanical systems [1].

Thus, for example, in calculating the kinetic momentum and kinetic energy, we need to know certain values that characterize the distribution of masses in the system. These values are: moments of inertia relative to a given point *O*, axis *ℓ* and coordinate axes *Oxyz*, as well as centripetal moments of inertia. These values are generally used in connection with the stability of rotating mechanical structures and other mechanical characteristics, as well as in solving some technical problems with minimum energy consumption, etc.

Methods of calculation of moments of inertia of some bodies are known, but their extreme properties are not studied. For this reason, let us consider the following problems.

Problem 1. Let there be a homogeneous material of mass *m* with density σ. What plane shape should be given to this material and how should it be placed on the *Oxy* plane so that its moment of inertia relative to the *Oz* axis takes extreme values, i.e. minimum or maximum value.

1. Let us first consider the question of the minimum moment of inertia.

First of all, let us show that if such a figure exists, its center of mass is at the origin *O* of the coordinate axes. Indeed, suppose that the center of mass *C* of this figure does not coincide with the origin *O*. Then according to the Huygens-Steiner theorem [2],

$$I_{oz} = I_{cz} + md^2, \tag{1}$$

where *d* is the distance between *O* and *C* points. It follows from (1) that $I_{oz} > I_{cz}$, since $d \neq 0$, what contradicts our assumption.

Now let us seek the plane figure among convex figures (or in the class of star-shaped plane figures relative to the center of mass).

Theorem 1. The moment of inertia between figures convex to the center of masses relative to the axis *Oz* reaches a minimum value when it has the shape of a circle centered at the point *O* and radius $r = \sqrt{\dfrac{m}{\pi\sigma}}$.

Proof. On the plane *Oxy* consider figure *G*, convex relative to the center of masses, whose center of masses is located at the origin of the coordinate axes and whose area is

$$S = \frac{m}{\sigma}. \tag{2}$$

As *G* is a convex, its center of masses *O* necessarily belongs to this area, i.e. $O \in G$. In this case, the boundary of area *G* will be written with polar coordinates as follows (Fig. 1)

$$\partial G : r = \rho(\varphi),\ 0 \le \varphi \le 2\pi, \tag{3}$$

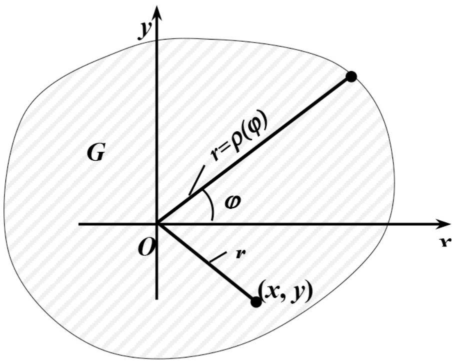

Fig. 1. A plane convex figure.

where $\rho = \rho(\varphi) \in C\,[0,\ 2\pi]$. Moreover, due to convexity, as it is known, function $\rho = \rho(\varphi)$ satisfies the Lipschitz condition on segment [0, 2π] [3]. From the analysis it is known that *G* is a figure whose boundary ∂G is given in polar coordinates by equation (3) calculated by the following formula [4]:

$$S = \frac{1}{2}\int_0^{2\pi} \rho^2(\varphi)d\varphi. \tag{4}$$

Let us denote the distance by $r = r(x, y) = \sqrt{x^2 + y^2}$ from point $(x, y) \in G$ to point *O*. After defining the moment of inertia of the figure *G* relative to the *Oz* axis, we will gain [2]

$$I_{oz} = \iint_G r^2(x,y)\sigma dxdy = \sigma\iint_G r^2(x,y)dxdy = \sigma\int_0^{2\pi} d\varphi \int_0^{\rho(\varphi)} r^3 dr =$$

$$= \sigma\int_0^{2\pi} d\varphi \left.\frac{r^4}{4}\right|_0^{\rho(\varphi)} = \frac{\sigma}{4}\int_0^{2\pi} \rho^4(\varphi)d\varphi. \tag{5}$$

Below we need the well-known Cauchy-Bunyakovsky inequality [5]

$$\left|\int_a^b f(x)g(x)dx\right|^2 \le \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx\,, \tag{6}$$

which is valid for any $f, g \in L_2[a,b]$ functions, in particular $\forall f, g \in C[a,b]$. It is also known that the equality in inequality (6) holds if and only if these functions are linearly dependent, i.e.

$$g(x) = cf(x),\ a \le x \le b,\ c = \text{const.} \tag{7}$$

By virtue of (6), we will gain

$$\int_0^{2\pi} 1\cdot \rho^2(\varphi)d\varphi \le \left(\int_0^{2\pi} 1^2 d\varphi\right)^{\frac{1}{2}} \cdot \left(\int_0^{2\pi} \rho^4(\varphi)d\varphi\right)^{\frac{1}{2}}, \tag{8}$$

from where we obtain

$$\frac{1}{2\pi}\left(\int_0^{2\pi} \rho^2(\varphi)d\varphi\right)^2 \le \int_0^{2\pi} \rho^4(\varphi)d\varphi\,. \tag{9}$$

It follows from (6), (7) and (8) that the equality in (9) holds then and only when

$$\rho^2(\varphi) \equiv const\,,\ 0 \le \varphi \le 2\pi. \tag{10}$$

Now from (2), (4), (5) and (9) it follows that

$$I_{oz} = \frac{\sigma}{4}\int_0^{2\pi}\rho^4(\varphi)d\varphi \ge \frac{\sigma}{8\pi}\left(\int_0^{2\pi}\rho^2(\varphi)d\varphi\right)^2 = \frac{\sigma}{8\pi}(2S)^2 = \frac{\sigma}{2\pi}\cdot S^2 = \frac{\sigma}{2\pi}\cdot\frac{m^2}{\sigma^2} = \frac{m^2}{2\pi\sigma}\,.$$

i.e.

$$I_{oz} \ge \frac{m^2}{2\pi\sigma} = \frac{1}{2}m\cdot\frac{m}{\pi\sigma} = I_{oz}^0\,, \tag{11}$$

where I_{oz}^0 is the moment of inertia of a circle $G_0 : x^2 + y^2 < \frac{m}{\pi\sigma}$ of homogenous mass m (with homogenous density σ) and radius $r = \sqrt{\frac{m}{\pi\sigma}}$ relative to the Oz axis.

It follows from (9) and (10) that the equality in inequality (11) holds if and only if

$$\rho(\varphi) \equiv const\,,$$

i.e. when the figure G is a circle, and since its area is $\frac{m}{\sigma}$, then

$$\rho(\varphi) \equiv r = \sqrt{\frac{m}{\pi\sigma}}\,.$$

That is what we wanted to prove.

This question is considered in the same way relative to the center of masses in the class of plane star-shaped figures.

2. Now let us consider the question of the maximum of the moment of inertia.

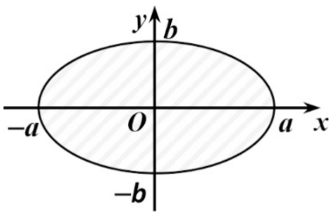

Fig. 2. Ellipse

let us consider an ellipse on the Oxy plane (Fig. 2):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

whose area is $S = \frac{m}{\sigma} = \pi ab$. As we know, for an ellipse [2]:

$$I_{oz} = \frac{1}{4}m(a^2 + b^2)\,.$$

Now if we take $b = \varepsilon$, and $a = \frac{S}{\pi b} = \frac{m}{\sigma\pi\varepsilon}$, then we will gain

$$\lim_{\varepsilon\to 0} I_{oz} = \lim_{\varepsilon\to 0} \frac{1}{4} m\left(\varepsilon^2 + \frac{m^2}{\pi^2\sigma^2\varepsilon^2}\right) = +\infty .$$

Therefore, the moment of inertia of a homogenous plane mass *m* (with σ density) can take any large value.

3. Now let us consider the same question, but not relative to the *Oz* axis, but to the ℓ axis passing across point *O*, making an angle α ($\alpha \neq 0$, $\alpha \neq \pi$) relative to the *Oz* axis. Without limiting generality, we can assume that the axis ℓ is located in the first and third quadrant of plane *Oyz*, i.e. $0 < \alpha < \frac{\pi}{2}$ (Fig. 3).

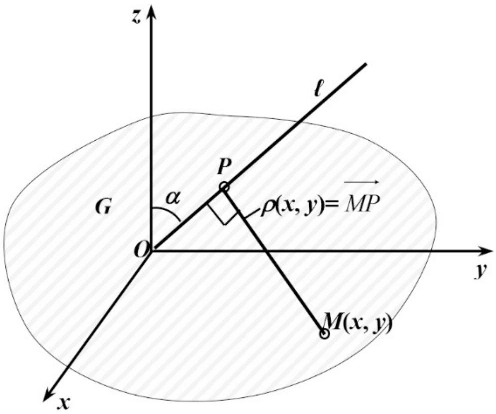

Fig. 3. Calculation of moment of inertia relative to axis ℓ.

Let us write the equation of the ℓ axis in parametric form:

$\ell : x = 0$, $y = kt$, $z = t$, $-\infty < t < +\infty$, where $k = \mathrm{tg}\alpha$.

Suppose $M(x, y) \in G$, and let us drop a *MP* normal from this point to the ℓ axis. If $P = (0,\ kt_0,\ t_0)$, then

$$OP \perp MP \Leftrightarrow \overrightarrow{OP}\cdot\overrightarrow{MP} = 0 \Leftrightarrow 0\cdot x + kt_0(y - kt_0) - t_0^2 = 0 \Rightarrow$$

$$\Rightarrow t_0 = \frac{ky}{1+k^2} \Rightarrow \rho^2(x, y) = \overrightarrow{MP}^2 = x^2 + \frac{y^2}{1+k^2},$$

i.e.
$$\rho^2(x, y) = x^2 + \frac{y^2}{1+k^2}. \tag{12}$$

By considering (12), we will gain

$$I_{o\ell} = \iint_G \rho^2(x, y)\sigma dxdy = \sigma\iint_G \left(x^2 + \frac{y^2}{1+k^2}\right)dxdy . \tag{13}$$

After linear transformation

$$\tilde{x} = x,\ \tilde{y} = \frac{1}{\sqrt{1+k^2}}y, \tag{14}$$

whose Jacobian is $\frac{\partial(x, y)}{\partial(\tilde{x}, \tilde{y})} = \sqrt{1+k^2}$, then *G* area will transfer into any $\tilde{G}$ area in the $O\tilde{x}\tilde{y}$ plane, while integral (13) will be rewritten as follows

$$I_{o\ell} = \tilde{\sigma}\iint_{\tilde{G}} (\tilde{x}^2 + \tilde{y}^2)d\tilde{x}d\tilde{y}, \tag{15}$$

where $\tilde{\sigma} = \sigma\sqrt{1+k^2}$.

By virtue of (5) and (15), this problem reduces to the previous problem, considered only on the $O\tilde{x}\tilde{y}$ plane for a plane convex figure $\tilde{G}$ with *m* mass and $\tilde{\sigma} = \sigma\sqrt{1+k^2}$ density.

Consequently, by virtue of the previous results $I_{o\ell}$ the moment of inertia reaches its minimum $\frac{m^2}{2\pi\tilde{\sigma}} = \frac{m^2}{2\pi\sigma\sqrt{1+k^2}}$, when $\tilde{G}$ area is a circle with its center at point *O* and radius

$$\tilde{R} = \sqrt{\frac{m}{\pi\tilde{\sigma}}} = \sqrt{\frac{m}{\pi\sigma\sqrt{1+k^2}}},$$

i.e.

$$\partial\tilde{G} : \tilde{x}^2 + \tilde{y}^2 = \frac{m}{\pi\sigma\sqrt{1+k^2}}. \tag{16}$$

Now, if we return to the variables x and y (14) according to equations (14), we obtain that the area G is an ellipse

$$\partial G : x^2 + \frac{y^2}{1+k^2} = \frac{m}{\pi\sigma\sqrt{1+k^2}},$$

i.e.

$$\partial G : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{17}$$

where $a = \sqrt{\frac{m}{\pi\sigma\sqrt{1+k^2}}}$, $b = \sqrt{\frac{m\sqrt{1+k^2}}{\pi\sigma}}$.

$$\min I_{o\ell} = \frac{m^2}{2\pi\sqrt{1+k^2}\cdot\sigma} = \frac{m^2}{2\pi\sqrt{1+tg^2\alpha}\cdot\sigma} = \frac{m^2\cos\alpha}{2\pi\sigma}. \tag{18}$$

It follows from (18) that

$$\lim_{\alpha\to\frac{\pi}{2}} \min I_{o\ell} = 0 \text{, i.e.} \qquad \min I_{o\ell\left(\alpha\to\frac{\pi}{2}\right)} = 0.$$

$$(17) \Rightarrow\ a = \sqrt{\frac{m}{\pi\sigma}}\cdot\sqrt{\cos\alpha},\ b = \sqrt{\frac{m}{\pi\sigma}}\cdot\frac{1}{\sqrt{\cos\alpha}} \text{ (Fig. 4).}$$

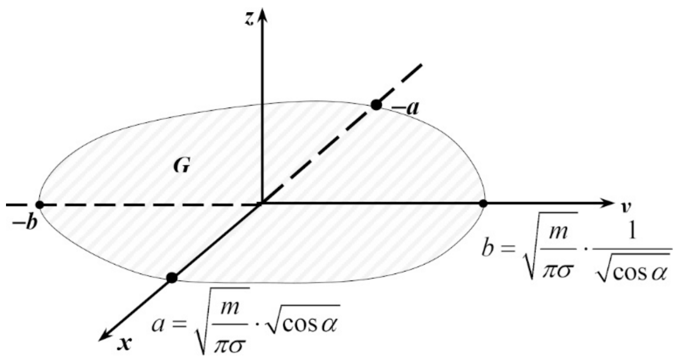

Fig. 4. An ellipse with *a* and *b* semi-axes.

4. Now let us consider the question of extreme properties of moments of inertia in a three dimensional case.

Problem 2. Suppose we have homogeneous material of mass m with density σ. What shape should be given to this material and how to arrange it in *Oxyz* space so that its polar moment of inertia with respect to the pole O takes extreme values, i.e., the minimum or the maximum value.

1°. (A minimum value case) It is easy to deduce from the Huygens-Steiner theorem that if such a body exists, its center of masses coincides with the pole O.

Now let us look for a sought figure (as a three-dimensional area) – among the star-shaped bodies relative to the center of masses. Let us denote a class of bodies by W_0. As it is known, convex bodies belong to class W_0.

Theorem 2. The polar moment of inertia of a homogenous material with m mass and σ density from class W_0 relative to the pole O reaches its minimum on the core with the center at the point O and radius $R=\sqrt[3]{\frac{3}{4}\cdot\frac{m}{\pi\sigma}}$.

Proof. Let us consider a star-shaped G body in the $Oxyz$ space relative to the O center of masses (Fig. 5), with its volume (r, θ, φ) in spherical coordinates is known to be calculated by the following formula

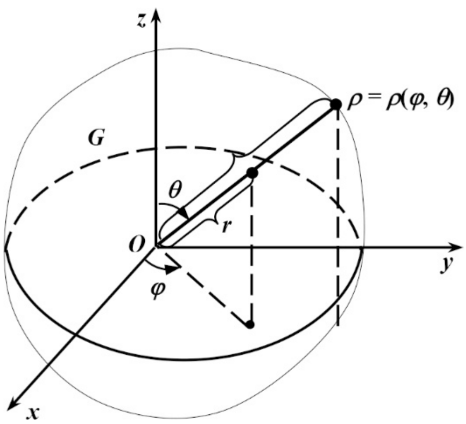

Fig. 5. Convex figure.

$$\frac{m}{\sigma}=V=\iiint_G 1\cdot dxdydz=\int_0^{\pi}d\theta\int_0^{2\pi}d\varphi\int_0^{\rho(\varphi,\theta)}r^2\sin\theta dr=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\pi}\frac{\rho^3(\varphi,\theta)}{3}\sin\theta d\theta=\frac{1}{3}\iint_{\Pi_0}\rho^3(\varphi,\theta)\sin\theta d\varphi d\theta, \quad (19)$$

where $\Pi_0=\{(\varphi,\theta)\in R^2: 0<\varphi<2\pi,\ 0<\theta<\pi\}$. In his case we used the fact that if $G\in W_0$, then its boundary ∂G is presented in polar coordinates with the following equation: $\rho=\rho(\varphi,\theta)$, $0\le\varphi\le 2\pi,\ 0\le\theta\le\pi$, where $\rho=\rho(\varphi,\theta)$ is a generally measurable and bounded function.

The polar moment of inertia of G body relative to O pole is defined by the following formula

$$I_0=\sigma\iiint_G r^2(x,y,z)dxdydz=\sigma\int_0^{\pi}d\theta\int_0^{2\pi}d\varphi\int_0^{\rho(\varphi,\theta)}r^4\sin\theta dr=$$

$$=\frac{\sigma}{5}\int_0^{\pi}d\theta\int_0^{2\pi}\rho^5(\varphi,\theta)\sin\theta d\varphi=\frac{\sigma}{5}\iint_{\Pi_0}\rho^5(\varphi,\theta)\sin\theta d\varphi d\theta. \quad (20)$$

We need generalized Holder's inequality, which is as follows [5]:

$$\iint_{\Pi_0}\left|f(\varphi,\theta)\cdot g(\varphi,\theta)\sin\theta\right|d\varphi d\theta\le\left(\iint_{\Pi_0}|f|^p\sin\theta d\Pi_0\right)^{\frac{1}{p}}\cdot\left(\iint_{\Pi_0}|g|^q\sin\theta d\Pi_0\right)^{\frac{1}{q}}, \quad (21)$$

where $\frac{1}{p}+\frac{1}{q}=1$, $(d\Pi_0=d\varphi d\theta)$, which holds for any $f\in L_p(\Pi_0,d\mu)$ functions, $g\in L_q(\Pi_0,d\mu)$, $d\mu=\sin\theta d\varphi d\theta$ is a positive measure. In particular, (21) is true $\forall f,g\in C(\Pi_0)$. It is also known that the equality in inequality (21) holds if and only if

$$\left|f(\varphi,\theta)\right|^p\equiv c\left|g(\varphi,\theta)\right|^q,\ c=const\ge 0,\ \forall(\varphi,\theta)\in\Pi_0. \quad (22)$$

By virtue of (19) and (21), where $p=\frac{5}{2}$, $q=\frac{5}{3}$ we will gain:

$$3\cdot\frac{m}{\sigma}=3V=\iint_{\Pi_0}\rho^3(\varphi,\theta)\sin\theta d\varphi d\theta=\iint_{\Pi_0}1\cdot\rho^3(\varphi,\theta)\sin\theta d\varphi d\theta\le$$

$$\le\left(\iint_{\Pi_0}1^p\cdot\sin\theta d\Pi_0\right)^{\frac{1}{p}}\cdot\left(\iint_{\Pi_0}\rho^{3q}\cdot(\varphi,\theta)\sin\theta d\varphi d\theta\right)^{\frac{1}{q}}=$$

$$=(4\pi)^{\frac{2}{5}}\left(\iint_{\Pi_0}\rho^5(\varphi,\theta)\sin\theta d\varphi d\theta\right)^{\frac{3}{5}}, \tag{23}$$

Wherefrom, by using (19) and (20), we obtain

$$I_0=\frac{\sigma}{5}\iint_{\Pi_0}\rho^5(\varphi,\theta)\sin\theta d\varphi d\theta\ge\frac{\sigma}{5}\left[\frac{1}{(4\pi)^{2/5}}\iint_{\Pi_0}\rho^3(\varphi,\theta)\sin\theta d\varphi d\theta\right]^{5/3}=$$

$$=\frac{\sigma}{5}\left[\frac{1}{(4\pi)^{2/5}}\cdot\frac{3m}{\sigma}\right]^{5/3}=\frac{\sigma}{5}\cdot\frac{1}{(4\pi)^{2/3}}\cdot\frac{(3m)^{5/3}}{\sigma^{5/3}}=\frac{1}{5(4\pi)^{2/3}}\cdot\frac{(3m)^{5/3}}{\sigma^{2/3}}. \tag{24}$$

From (24) it follows

$$\min I_0=\frac{1}{5(4\pi)^{2/3}}\cdot\frac{(3m)^{5/3}}{\sigma^{2/3}},$$

and by virtue of (22) and (23), this minimum is reached on the body for which $1^p\equiv c\rho^3(\varphi,\theta)$, i.e. $\rho(\varphi,\theta)\equiv const$ and we are dealing with a core with radius $R=\sqrt[3]{\frac{3}{4}\cdot\frac{m}{\pi\sigma}}$. This is what we wanted to prove.

მექანიკა

მატერიალური სხეულების ინერციის მომენტების ექსტრემალური თვისებების შესახებ

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განხილულია ამოცანა, სადაც მოცემულია m მასის ერთგვაროვანი მასალა, რომლის სიმკვრივეა σ. საძიებელია, რა სახის ბრტყელი ფორმა უნდა ჰქონდეს ამ მასალას და Oxy სიბრტყეზე როგორ უნდა იყოს განლაგებული, რომ მისმა ინერციის მომენტმა Oz ღერძის მიმართ მიიღოს ექსტრემალური მნიშვნელობები. ამოცანის ამოხსნის შედეგად მიღებულია, რომ ამოზნექილ ვარსკვლავისებრ ბრტყელ ფიგურათა შორის ინერციის მომენტი Oz ღერძის მიმართ აღწევს თავის მინიმალურ მნიშვნელობას, როცა მას აქვს წრის ფორმა ცენტრით O წერტილში და რადიუსით r. როცა ბრტყელ ფიგურას აქვს ელიფსის ფორმა, მაშინ Oz ღერძის მიმართ ინერციის მომენტმა შეიძლება მიაღწიოს რაგინდ დიდ მნიშვნელობას. განხილულია იგივე ამოცანა, როცა ინერციის მომენტი გამოთვლილია მასათა ცენტრზე გამავალი ნებისმიერი $O\ell$ ღერძის მიმართ, რომელიც Oz ღერძთან შეადგენს α ($\alpha \neq 0,\ \alpha \neq \pi$) კუთხეს. ამ შემთხვევაში ინერციის მომენტი აღწევს თავის მინიმალურ მნიშვნელობას, როცა ბრტყელ ფიგურას აქვს ელიფსის ფორმა a და b ნახევარღერძებით. სამგანზომილებიან შემთხვევაში m მასისა და σ სიმკვრივის მასალის პოლარული ინერციის მომენტი O პოლუსის მიმართ აღწევს თავის მინიმუმს, როცა მასალას აქვს ბირთვის ფორმა O ცენტრით და R რადიუსით.

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