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Mathematics

On the Approximation of a Nonclassical Three-Dimensional Model with Three Phase-Lags for Thermoelastic Plates by Two-Dimensional Problems

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In the present paper, a nonclassical dynamic model with three phase-lags for a thermoelastic plate with variable thickness, which may vanish on a part of the lateral boundary, is studied by applying variational approach. An algorithm of approximation of the three-dimensional model by a sequence of two-dimensional problems is constructed, when the densities of surface force and heat flux along the outward normal vector of the boundary are given on the plate's upper and lower face surfaces. The constructed two-dimensional initial-boundary value problems are investigated in suitable function spaces, the convergence in the corresponding spaces of the sequence of vector-functions of three space variables, restored from the solutions of two-dimensional problems, to the solution of the original three-dimensional problem is proved and, under additional conditions, the rate of convergence is estimated. *© 2024 Bull. Georg. Natl. Acad. Sci.*

nonclassical thermoelasticity, plate with variable thickness, approximation algorithm

The classical Fourier law of heat conduction is unsuitable for the successful description of thermal shocks observed in various modern technological processes, which are caused by extremely high temperature gradients and very short operation periods of the order of picoseconds [1-3]. Two first nonclassical models, which were developed to eliminate the unrealistic behaviour of the classical model for thermoelastic solids, were proposed by Lord and Shulman [4] and Green and Lindsay [5]. Later on, Tzou [6] proposed a generalization of Fourier's law with two phase-lags and by applying it Chandrasekharaiah [7] constructed a nonclassical model for thermoelastic solids, which is an extension of the Lord-Shulman nonclassical model for thermoelastic bodies. Further, by using the Green-Naghdy model III [8] of heat flow Roy Choudhuri [9] proposed a generalization of the dual-phase-lag heat conduction model by Tzou, which includes not only the temperature gradient but also the thermal displacement gradient and depends on three

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phase-lags for the heat flux vector, the temperature gradient and the thermal displacement gradient. The initial-boundary value problem with mixed boundary conditions corresponding to the Roy Choudhuri dynamic three-dimensional model with three phase-lags for thermoelastic solids consisting of anisotropic homogeneous material was investigated in [10], and problems of propagation of waves, thermodynamic compatibility, uniqueness and continuous dependence problems, methods of numerical solution of corresponding problems and related topics are considered by many researchers (see [11-14] and the references given therein).

The present paper is devoted to the construction and investigation of an algorithm of approximation of the nonclassical model with three phase-lags proposed by Roy Choudhuri for thermoelastic plates by a hierarchy of two-dimensional problems. To construct an approximation algorithm, we use generalization and extension of the dimensional reduction method proposed by I. Vekua [15] for homogeneous isotropic plates in the classical theory of elasticity. By applying a variational formulation of the three-dimensional initial-boundary value problem for a thermoelastic plate with variable thickness, which may vanish on a part of the lateral boundary, when the densities of surface force and heat flux along the outward normal vector of the boundary are given along the plate's upper and lower face surfaces, we construct a hierarchy of two-dimensional problems. We investigate the existence and uniqueness of solutions of the constructed two-dimensional problems in suitable function spaces. Moreover, we prove that the sequence of vector-functions of three space variables restored from the solutions of the two-dimensional problems converges in corresponding spaces to the solution of the original three-dimensional problem and, if it possesses additional regularity, we estimate the rate of convergence.

We denote by $W^{r,2}(D)=H^r(D)$ and $H^{\hat{r}}(\hat{\Gamma})$, $r,\hat{r}\in\mathbf{R}$, $r\geq 0$, $0\leq\hat{r}\leq 1$, the Sobolev spaces of orders $r,\hat{r}$ based on the spaces $H^0(D)=L^2(D)$ and $H^0(\hat{\Gamma})=L^2(\hat{\Gamma})$ of square-integrable functions, respectively, where $D\subset\mathbf{R}^p$, $p\in\mathbf{N}$, is a bounded Lipschitz domain [16] and $\hat{\Gamma}\subset\partial D$ is a Lipschitz surface. We denote by $\mathbf{H}^r(D)=[H^r(D)]^3$, $\mathbf{H}^{\hat{r}}(\hat{\Gamma})=[H^{\hat{r}}(\hat{\Gamma})]^3$, $\mathbf{L}^2(D)=[L^2(D)]^3$, $\mathbf{L}^s(\hat{\Gamma})=[L^s(\hat{\Gamma})]^3$, $r\geq 0$, $0\leq\hat{r}\leq 1$, $s\geq 1$, $r,\hat{r},s\in\mathbf{R}$, the corresponding spaces of vector-valued functions. The trace operators are denoted by $tr_{\hat{\Gamma}}:H^1(D)\to H^{1/2}(\hat{\Gamma})$ and $\mathbf{tr}_{\hat{\Gamma}}:\mathbf{H}^1(D)\to\mathbf{H}^{1/2}(\hat{\Gamma})$. For a Banach space X, we denote by $C([0,T];X)$ the space of continuous functions on $[0,T]$ with values in X. $L^q(0,T;X)$, $1\leq q\leq\infty$, is the space of such measurable functions $g:(0,T)\to X$ that $\|g(t)\|_X\in L^q(0,T)$ and the generalized first, second, third and arbitrary $k\in\mathbf{N}$-th order derivatives of g are denoted by $g'=dg/dt$, $g''=d^2g/dt^2$, $g'''=d^3g/dt^3$ and $g^{(k)}=d^kg/dt^k$ [17].

Let us consider thermoelastic plate with variable thickness, which may vanish on a part of its boundary, i.e. the initial configuration $\overline{\Omega}$ of the plate is a closure of the following Lipschitz domain

$$\Omega=\{(x_1,x_2,x_3)\in\mathbf{R}^3;\, h^-(x_1,x_2)<x_3<h^+(x_1,x_2),\quad (x_1,x_2)\in\omega\},$$

where $\omega\subset\mathbf{R}^2$ is a two-dimensional bounded Lipschitz domain with boundary $\partial\omega$, $h^{\pm}\in C^0(\overline{\omega})\cap C^{4,1}_{loc}(\omega\cup\tilde{\gamma})$ are Lipschitz continuous in the interior of the domain ω and on $\tilde{\gamma}\subset\partial\omega$ together with their derivatives up to the fourth order, $h^+(x_1,x_2)>h^-(x_1,x_2)$, for $(x_1,x_2)\in\omega\cup\tilde{\gamma}$, $\tilde{\gamma}\subset\partial\omega$ is a Lipschitz curve, $h^+(x_1,x_2)=h^-(x_1,x_2)$, for $(x_1,x_2)\in\partial\omega\setminus\tilde{\gamma}$. The upper and the lower face surfaces of Ω, defined by the equations $x_3 = h^+(x_1,x_2)$ and $x_3 = h^-(x_1,x_2)$, $(x_1,x_2)\in\omega$, we denote by Γ^+ and Γ^-, respectively, and the lateral surface, where the thickness of Ω is positive, we denote by $\tilde\Gamma = \partial\Omega \setminus \overline{(\Gamma^+\cup\Gamma^-)} = \{(x_1,x_2,x_3)\in\mathbf{R}^3; h^-(x_1,x_2)<x_3<h^+(x_1,x_2),(x_1,x_2)\in\tilde\gamma\}$. The plate is subjected to applied body force with density $\mathbf{f}=(f_i):\Omega\times(0,T)\to\mathbf{R}^3$ and heat source with density $\tilde f^\theta:\Omega\times(0,T)\to\mathbf{R}$. The plate is clamped and the temperature θ vanishes along a part $\tilde\Gamma_0=\{(x_1,x_2,x_3)\in\mathbf{R}^3;\ h^-(x_1,x_2)<x_3<h^+(x_1,x_2),$ $(x_1,x_2)\in\tilde\gamma_0\}$ of the lateral surface $\tilde\Gamma$, $\tilde\gamma_0\subset\tilde\gamma$ is a Lipschitz curve, and on the remaining part $\Gamma_1=\Gamma\setminus\overline{\tilde\Gamma_0}$ of the boundary the density of the surface force $\mathbf{g}=(g_i):\Gamma_1\times(0,T)\to\mathbf{R}^3$ and the density of heat flux $\tilde g^\theta:\Gamma_1\times(0,T)\to\mathbf{R}$ along the outward normal vector of Γ are given.

The nonclassical dynamic linear three-dimensional model with three phase-lags for the thermoelastic plate $\overline\Omega$ consisting of homogeneous isotropic material is given by the following initial-boundary value problem in differential form [9, 10]:

$$\rho\frac{\partial^2 u_i}{\partial t^2}-\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\lambda\sum_{p=1}^{3}e_{pp}(\mathbf{u})\delta_{ij}+2\mu e_{ij}(\mathbf{u})+\eta\theta\delta_{ij}\right)=f_i \quad \text{in } \Omega\times(0,T),\quad i=1,2,3, \tag{1}$$

$$\chi\left(\frac{\partial^2\theta}{\partial t^2}+\tau_0\frac{\partial^3\theta}{\partial t^3}+\frac{\tau_0^2}{2}\frac{\partial^4\theta}{\partial t^4}\right)-\kappa\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\left(\frac{\partial\theta}{\partial t}+\tau_1\frac{\partial^2\theta}{\partial t^2}\right)-\bar\kappa\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\left(\theta+\tau_2\frac{\partial\theta}{\partial t}\right)-$$

$$-\Theta_0\eta\sum_{p=1}^{3}e_{pp}\left(\frac{\partial^2\mathbf{u}}{\partial t^2}+\tau_0\frac{\partial^3\mathbf{u}}{\partial t^3}+\frac{\tau_0^2}{2}\frac{\partial^4\mathbf{u}}{\partial t^4}\right)=\frac{\partial\tilde f^\theta}{\partial t}+\tau_0\frac{\partial^2\tilde f^\theta}{\partial t^2}+\frac{\tau_0^2}{2}\frac{\partial^3\tilde f^\theta}{\partial t^3} \quad \text{in } \Omega\times(0,T), \tag{2}$$

$$\mathbf{u}=\mathbf{0}\quad \text{on } \Gamma_0\times(0,T),\quad \sum_{j=1}^{3}\left(\lambda\sum_{p=1}^{3}e_{pp}(\mathbf{u})\delta_{ij}+2\mu e_{ij}(\mathbf{u})+\eta\theta\delta_{ij}\right)\nu_j=g_i \quad \text{on } \Gamma_1\times(0,T),\ i=1,2,3, \tag{3}$$

$$\theta=0\quad \text{on } \Gamma_0\times(0,T),\ -\kappa\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial\theta}{\partial t}+\tau_1\frac{\partial^2\theta}{\partial t^2}\right)\nu_j-\bar\kappa\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\theta+\tau_2\frac{\partial\theta}{\partial t}\right)\nu_j=\frac{\partial\tilde g^\theta}{\partial t}\quad \text{on } \Gamma_1\times(0,T), \tag{4}$$

$$\mathbf{u}(x,0)=\mathbf{u}_0(x),\quad \frac{\partial\mathbf{u}}{\partial t}(x,0)=\mathbf{u}_1(x),\qquad x\in\Omega, \tag{5}$$

$$\theta(x,0)=\theta_0(x),\quad \frac{\partial\theta}{\partial t}(x,0)=\theta_1(x),\quad \frac{\partial^2\theta}{\partial t^2}(x,0)=\theta_2(x),\quad \frac{\partial^3\theta}{\partial t^3}(x,0)=\theta_3(x),\qquad x\in\Omega, \tag{6}$$

where δ_{ij} is the Kronecker's delta, $e_{ij}(\mathbf{v})=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)$, $i,j=1,2,3$, $\mathbf{v}=(v_i)_{i=1}^3:\Omega\to\mathbf{R}^3$, is the linearized strain tensor, $\mathbf{v}=(\nu_i)_{i=1}^3$ is the unit outward normal to Γ, $\mathbf{u}=(u_i)_{i=1}^3:\Omega\times(0,T)\to\mathbf{R}^3$ is the displacement vector-function of the plate, $\theta:\Omega\times(0,T)\to\mathbf{R}$ is the temperature distribution, $\mathbf{u}_0=(u_{0i})_{i=1}^3$ and $\mathbf{u}_1=(u_{1i})_{i=1}^3$ are the initial displacement and velocity vector-functions, and θ_0 is the initial distribution of temperature, θ_1, θ_2 and θ_3 are the rate of change, the second and the third order derivatives of temperature with respect to the time variable at the initial moment of time, ρ is the mass density in the reference configuration, λ,μ are the Lamé constants, η is the stress-temperature coefficient, $\chi>0$ is the volumetric heat capacity, $\kappa>0$ is the thermal conductivity coefficient and $\bar\kappa>0$ is the Green-Naghdi thermal conductivity rate coefficient [8], $\Theta_0>0$ is the temperature of the thermoelastic body in natural state of no deformation, which is considered as a reference temperature, $\tau_0 \geq 0$, $\tau_1 \geq 0$ and $\tau_2 \geq 0$ stand for the heat flux, temperature gradient and thermal displacement gradient phase-lags, respectively.

To investigate the three-dimensional initial-boundary value problem (1)-(6) we consider the following variational formulation in the spaces of vector-valued distributions with respect to the time variable t, which is equivalent to the original differential formulation (1)-(6) in the spaces of classical smooth enough functions: Find the unknown vector-function $\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''' \in C([0,T];\mathbf{V}(\Omega))$, $\mathbf{u}^{(4)} \in L^2(0,T;\mathbf{V}(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{u}^{(5)} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\theta,\theta',\theta'' \in C([0,T];V^\theta(\Omega))$, $\theta''' \in L^2(0,T;V^\theta(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\theta^{(4)} \in L^2(0,T;L^2(\Omega))$, which satisfy the following equations in the sense of distributions on $(0,T)$,

$$\rho(\mathbf{u}'',\mathbf{v})_{\mathbf{L}^2(\Omega)} + a(\mathbf{u},\mathbf{v}) + b(\theta,\mathbf{v}) = (\mathbf{f},\mathbf{v})_{\mathbf{L}^2(\Omega)} + (\mathbf{g},\mathbf{tr}_{\Gamma_1}(\mathbf{v}))_{\mathbf{L}^2(\Gamma_1)}, \qquad \forall \mathbf{v} \in \mathbf{V}(\Omega), \tag{7}$$

$$\chi\left(\theta'' + \tau_0\theta''' + \frac{\tau_0^2}{2}\theta^{(4)},\varphi\right)_{L^2(\Omega)} + a_1^\theta(\theta' + \tau_1\theta'',\varphi) + a_2^\theta(\theta + \tau_2\theta',\varphi) -$$

$$-\Theta_0 b^\theta\left(\mathbf{u}'' + \tau_0\mathbf{u}''' + \frac{\tau_0^2}{2}\mathbf{u}^{(4)},\varphi\right) = (f^\theta,\varphi)_{L^2(\Omega)} - (g^\theta, tr_{\Gamma_1}(\varphi))_{L^2(\Gamma_1)}, \qquad \forall \varphi \in V^\theta(\Omega), \tag{8}$$

together with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0,\ \mathbf{u}'(0) = \mathbf{u}_1,\ \theta(0) = \theta_0,\ \theta'(0) = \theta_1,\ \theta''(0) = \theta_2,\ \theta'''(0) = \theta_3, \tag{9}$$

where $\mathbf{V}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega); \mathbf{tr}_\Gamma(\mathbf{v}) = \mathbf{0} \text{ on } \Gamma_0\}$, $V^\theta(\Omega) = \{\varphi \in H^1(\Omega); tr_\Gamma(\varphi) = 0 \text{ on } \Gamma_0\}$,

$$a(\tilde{\mathbf{v}},\mathbf{v}) = \int_\Omega \left(\lambda\sum_{p=1}^3 e_{pp}(\tilde{\mathbf{v}})\sum_{q=1}^3 e_{qq}(\mathbf{v}) + 2\mu\sum_{i,j=1}^3 e_{ij}(\tilde{\mathbf{v}})e_{ij}(\mathbf{v})\right)dx, \qquad \forall \mathbf{v},\tilde{\mathbf{v}} \in \mathbf{H}^1(\Omega),$$

$$a_1^\theta(\tilde{\varphi},\varphi) = \kappa\int_\Omega \sum_{j=1}^3 \frac{\partial\tilde{\varphi}}{\partial x_j}\frac{\partial\varphi}{\partial x_j}dx, \quad a_2^\theta(\tilde{\varphi},\varphi) = \bar{\kappa}\int_\Omega \sum_{j=1}^3 \frac{\partial\tilde{\varphi}}{\partial x_j}\frac{\partial\varphi}{\partial x_j}dx, \qquad \forall \varphi,\tilde{\varphi} \in H^1(\Omega),$$

$$b(\varphi,\mathbf{v}) = b^\theta(\mathbf{v},\varphi) = \eta\int_\Omega \varphi\sum_{p=1}^3 e_{pp}(\mathbf{v})dx, \qquad \forall \varphi \in L^2(\Omega), \mathbf{v} \in \mathbf{H}^1(\Omega),$$

$f^\theta = \dfrac{\partial\tilde{f}^\theta}{\partial t} + \tau_0\dfrac{\partial^2\tilde{f}^\theta}{\partial t^2} + \dfrac{\tau_0^2}{2}\dfrac{\partial^3\tilde{f}^\theta}{\partial t^3}$, $g^\theta = \dfrac{\partial\tilde{g}^\theta}{\partial t}$, $(.,.)_{\mathbf{L}^2(\Omega)}$, $(.,.)_{L^2(\Omega)}$, $(.,.)_{\mathbf{L}^2(\Gamma_1)}$ and $(.,.)_{L^2(\Gamma_1)}$ are scalar products in the spaces $\mathbf{L}^2(\Omega)$, $L^2(\Omega)$, $\mathbf{L}^2(\Gamma_1)$ and $L^2(\Gamma_1)$, respectively.

For the nonclassical dynamic three-dimensional model (7)-(9) with three phase-lags the following existence and uniqueness theorem is valid [10].

Theorem 1. Suppose that $\Omega \subset \mathbf{R}^3$ is a bounded Lipschitz domain and $\rho > 0$, $\mu > 0$, $3\lambda + 2\mu > 0$, $\chi > 0$, $\kappa > 0$, $\bar{\kappa} > 0$, $\tau_0 > 0$, $\tau_1 > 0$, $\tau_2 > 0$. If $\mathbf{f} \in C([0,T];\mathbf{H}^3(\Omega))$, $\mathbf{f}' \in C([0,T];\mathbf{H}^2(\Omega))$, $\mathbf{f}'' \in C([0,T];\mathbf{H}^1(\Omega))$, $\mathbf{f}''',\mathbf{f}^{(4)} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g},\mathbf{g}',\mathbf{g}'',\mathbf{g}''',\mathbf{g}^{(4)},\mathbf{g}^{(5)} \in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $f^\theta, f^{\theta'} \in L^2(0,T;L^2(\Omega))$, $g^\theta, g^{\theta'}, g^{\theta''} \in L^2(0,T;L^{4/3}(\Gamma_1))$, and the initial conditions $\mathbf{u}_0 \in \mathbf{H}^5(\Omega) \cap \mathbf{V}(\Omega)$, $\mathbf{u}_1 \in \mathbf{H}^4(\Omega) \cap \mathbf{V}(\Omega)$, $\theta_0 \in H^4(\Omega) \cap V^\theta(\Omega)$, $\theta_1 \in H^3(\Omega) \cap V^\theta(\Omega)$, $\theta_2 \in H^2(\Omega) \cap V^\theta(\Omega)$, $\theta_3 \in V^\theta(\Omega)$, satisfy the following compatibility conditions

$$g_i^{(k)}(0)=tr_{\Gamma_1}\left(\sum_{j=1}^{3}\left(\lambda\sum_{p=1}^{3}e_{pp}(\mathbf{u}_k)\delta_{ij}+2\mu e_{ij}(\mathbf{u}_k)+\eta\theta_k\delta_{ij}\right)\nu_j\right),\quad k=0,1,2,3,\quad i=1,2,3, \tag{10}$$

$$g^{\theta}(0)=-tr_{\Gamma_1}\left(\sum_{j=1}^{3}\kappa\left(\frac{\partial\theta_1}{\partial x_j}+\tau_1\frac{\partial\theta_2}{\partial x_j}\right)\nu_j+\sum_{j=1}^{3}\overline{\kappa}\left(\frac{\partial\theta_0}{\partial x_j}+\tau_2\frac{\partial\theta_1}{\partial x_j}\right)\nu_j\right), \tag{11}$$

where $\mathbf{v}=(\nu_i)_{i=1}^3$ is the unit outward normal vector to Γ,

$$u_{\alpha+2\,i}=\frac{1}{\rho}\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\lambda\sum_{p=1}^{3}e_{pp}(\mathbf{u}_\alpha)\delta_{ij}+2\mu e_{ij}(\mathbf{u}_\alpha)+\eta\theta_\alpha\delta_{ij}\right)+f_i^{(\alpha)}(0)\right),\quad \alpha=0,1,\ i=1,2,3,$$

then the initial-boundary value problem (7)-(9) possesses a unique solution.

To construct an algorithm of approximation of the nonclassical three-dimensional model with three phase-lags for thermoelastic plate by a sequence of two-dimensional problems, let us consider the subspaces $\mathbf{V_N}(\Omega)\subset\mathbf{V}(\Omega)$, $\mathbf{H_N}(\Omega)\subset\mathbf{L}^2(\Omega)$, $\mathbf{V_N^k}(\Omega)\subset\mathbf{H}^k(\Omega)\cap\mathbf{V}(\Omega)$, $k=2,3,4,5$, $\mathbf{H_N^{\overline{k}}}(\Omega)\subset\mathbf{H}^{\overline{k}}(\Omega)$, $\overline{k}=1,2,3$, $\mathbf{N}=(N_1,N_2,N_3)$, consisting of vector-functions whose components are polynomials with respect to the variable x_3,

$$\mathbf{v_N}=(v_{\mathbf{N}i}),\quad v_{\mathbf{N}i}=\sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{v}_{\mathbf{N}i}\,P_{r_i}(z),\quad h^{-1/2}\,\overset{r_i}{v}_{\mathbf{N}i}\in L^2(\omega),\ \ 0\le r_i\le N_i,\ \ i=1,2,3,$$

where $z=\dfrac{x_3-\overline{h}}{h}$, $h=\dfrac{h^+-h^-}{2}$, $\overline{h}=\dfrac{h^++h^-}{2}$. In addition, we consider the subspaces $V_{N_\theta}^{\theta}(\Omega)\subset V^{\theta}(\Omega)$, $H_{N_\theta}^{\theta}(\Omega)\subset L^2(\Omega)$, $V_{N_\theta}^{\theta,k_\theta}(\Omega)\subset H^{k_\theta}(\Omega)\cap V^{\theta}(\Omega)$, $k_\theta=2,3,4$, which consist of the following functions

$$\varphi_{N_\theta}=\sum_{r=0}^{N_\theta}\frac{1}{h}(r+\frac{1}{2})\overset{r}{\varphi}_{N_\theta}\,P_r(z),\quad h^{-1/2}\,\overset{r}{\varphi}_{N_\theta}\in L^2(\omega),\ \ 0\le r\le N_\theta.$$

Since the functions h^+ and h^- are Lipschitz continuous together with their derivatives up to the fourth order in the interior of the domain ω, from Rademacher's theorem [18] it follows that $h^\pm$, $\partial_\alpha h^\pm$, $\partial_\alpha\partial_{\alpha_1}h^\pm$, $\partial_\alpha\partial_{\alpha_1}\partial_{\alpha_2}h^\pm$, $\partial_\alpha\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}h^\pm$ are differentiable almost everywhere in ω^* and $\partial_\alpha\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}\partial_{\alpha_4}h^\pm\in L^\infty(\omega^*)$ for all subdomains ω^*, $\overline{\omega^*}\subset\omega$, $\alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4=1,2$. Therefore, the positiveness of h in ω implies that for any vector-function $\mathbf{v_N}=(v_{\mathbf{N}i})_{i=1}^3\in\mathbf{V_N^k}(\Omega)$ and $\mathbf{v_N}\in\mathbf{V_N}(\Omega)$ the corresponding functions belong to $H^k(\omega^*)$ and $H^1(\omega^*)$, for all ω^*, $\overline{\omega^*}\subset\omega$, i.e. $\overset{r_i}{v}_{\mathbf{N}i}\in H_{loc}^k(\omega)$ and $\overset{r_i}{v}_{\mathbf{N}i}\in H_{loc}^1(\omega)$, $0\le r_i\le N_i$, $i=1,2,3$, $k=2,3,4,5$. For functions from the spaces $V_{N_\theta}^{\theta,k_\theta}(\Omega)$ and $V_{N_\theta}^{\theta}(\Omega)$ we also have $\overset{r}{\varphi}_{N_\theta}\in H_{loc}^{k_\theta}(\omega)$, if $\varphi_{N_\theta}\in V_{N_\theta}^{\theta,k_\theta}(\Omega)$, and $\overset{r}{\varphi}_{N_\theta}\in H_{loc}^1(\omega)$, if $\varphi_{N_\theta}\in V_{N_\theta}^{\theta}(\Omega)$, $0\le r\le N_\theta$, $k_\theta=2,3,4$. Moreover, the norms $\|.\|_{\mathbf{H}^k(\Omega)}$, $k=1,2,3,4,5$, and $\|.\|_{H^{k_\theta}(\Omega)}$, $k_\theta=1,2,3,4$, in the spaces $\mathbf{H}^k(\Omega)$ and $H^{k_\theta}(\Omega)$ define weighted norms $\|.\|_{k*}$ and $\|.\|_{\theta k_\theta *}$ of vector-functions $\vec{v}_{\mathbf{N}}\in[H_{loc}^k(\omega)]^{N_{1,2,3}}$, $N_{1,2,3}=N_1+N_2+N_3+3$, with components $\overset{r_i}{v}_{\mathbf{N}i}$, $\vec{v}_{\mathbf{N}}=(\overset{r_i}{v}_{\mathbf{N}i})$, and $\vec{\varphi}_{N_\theta}\in[H_{loc}^{k_\theta}(\omega)]^{N_\theta+1}$, with components $\overset{r}{\varphi}_{N_\theta}$, $\vec{\varphi}_{N_\theta}=(\overset{r}{\varphi}_{N_\theta})$, such that $\|\vec{v}_{\mathbf{N}}\|_{k*}=\|\mathbf{v_N}\|_{\mathbf{H}^k(\Omega)}$ and $\|\vec{\varphi}_{N_\theta}\|_{\theta k_\theta *}=\|\varphi_{N_\theta}\|_{H^{k_\theta}(\Omega)}$. Using the properties of the Legendre polynomials, we can obtain explicit expressions for the norms $\|.\|_{k*}$ and $\|.\|_{\theta k_\theta *}$

For components $\overset{r_i}{v}_{\mathbf{N}i}$ and $\overset{r}{\varphi}_{N_\theta}$ of vector-functions $\vec{v}_{\mathbf{N}} \in [H^1_{loc}(\omega)]^{N_{1,2,3}}$ and $\vec{\varphi}_{N_\theta} \in [H^1_{loc}(\omega)]^{N_\theta+1}$ that satisfy the conditions $\|\vec{v}_{\mathbf{N}}\|_{1*} < \infty$ and $\|\vec{\varphi}_{N_\theta}\|_{\theta 1*} < \infty$ we can define the traces on $\tilde{\gamma}$. Indeed, the corresponding vector-function $\mathbf{v}_{\mathbf{N}} = (v_{\mathbf{N}i})_{i=1}^3$ and function φ_{N_θ} of three space variables belong to the space $\mathbf{V}_{\mathbf{N}}(\Omega) \subset \mathbf{H}^1(\Omega)$ and $V^\theta_{N_\theta}(\Omega) \subset H^1(\Omega)$, respectively. Consequently, applying the trace operator on the space $H^1(\Omega)$ we define the traces on $\tilde{\gamma}$ for functions $\overset{r_i}{v}_{\mathbf{N}i}$ and $\overset{r}{\varphi}_{N_\theta}$,

$$tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i}) = \int_{h^-}^{h^+} tr_{\tilde{\Gamma}}(v_{\mathbf{N}i}) P_{r_i}(z) dx_3, \;\; tr_{\tilde{\gamma}}(\overset{r}{\varphi}_{N_\theta}) = \int_{h^-}^{h^+} tr_{\tilde{\Gamma}}(\varphi_{N_\theta}) P_r(z) dx_3, \;\; r_i = 0,\dots,N_i, i = 1,2,3, r = 0,\dots,N_\theta.$$

Since the vector-functions $\mathbf{v}_{\mathbf{N}} = (v_{\mathbf{N}i})_{i=1}^3$ from the subspaces $\mathbf{V}_{\mathbf{N}}(\Omega)$ and $\mathbf{H}_{\mathbf{N}}(\Omega)$, and the functions φ_{N_θ} from $V^\theta_{N_\theta}(\Omega)$ and $H^\theta_{N_\theta}(\Omega)$ are uniquely defined by functions $\overset{r_i}{v}_{\mathbf{N}i}$ and $\overset{r}{\varphi}_{N_\theta}$ of two space variables, considering the three-dimensional problem (7)-(9) on these subspaces, we obtain the following hierarchy of two-dimensional initial-boundary value problems: Find $\vec{w}_{\mathbf{N}}, \vec{w}'_{\mathbf{N}}, \vec{w}''_{\mathbf{N}}, \vec{w}'''_{\mathbf{N}} \in C([0,T]; \vec{V}_{\mathbf{N}}(\omega))$, $\vec{w}^{(4)}_{\mathbf{N}} \in L^2(0,T; \vec{V}_{\mathbf{N}}(\omega)) \cap L^\infty(0,T; \vec{H}_{\mathbf{N}}(\omega))$, $\vec{w}^{(5)}_{\mathbf{N}} \in L^2(0,T; \vec{H}_{\mathbf{N}}(\omega))$, $\vec{\zeta}_{N_\theta}, \vec{\zeta}'_{N_\theta}, \vec{\zeta}''_{N_\theta} \in C([0,T]; \vec{V}^\theta_{N_\theta}(\omega))$, $\vec{\zeta}'''_{N_\theta} \in L^2(0,T; \vec{V}^\theta_{N_\theta}(\omega)) \cap L^\infty(0,T; \vec{H}^\theta_{N_\theta}(\omega))$, $\vec{\zeta}^{(4)}_{N_\theta} \in L^2(0,T; \vec{H}^\theta_{N_\theta}(\omega))$, which satisfy the following equations in the sense of distributions on $(0,T)$,

$$R_{\mathbf{N}}(\vec{w}''_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) + a_{\mathbf{N}}(\vec{w}_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) + b_{\mathbf{N}N_\theta}(\vec{\zeta}_{N_\theta}, \vec{v}_{\mathbf{N}}) = L_{\mathbf{N}}(\vec{v}_{\mathbf{N}}), \qquad \forall \vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega), \tag{12}$$

$$R^\theta_{N_\theta}\left(\vec{\zeta}''_{N_\theta} + \tau_0 \vec{\zeta}'''_{N_\theta} + \frac{\tau_0^2}{2}\vec{\zeta}^{(4)}_{N_\theta}, \vec{\varphi}_{N_\theta}\right) + a^\theta_{1N_\theta}(\vec{\zeta}'_{N_\theta} + \tau_1 \vec{\zeta}''_{N_\theta}, \vec{\varphi}_{N_\theta}) + a^\theta_{2N_\theta}(\vec{\zeta}_{N_\theta} + \tau_2 \vec{\zeta}'_{N_\theta}, \vec{\varphi}_{N_\theta}) -$$
$$-\Theta_0 b^\theta_{\mathbf{N}N_\theta}\left(\vec{w}''_{\mathbf{N}} + \tau_0 \vec{w}'''_{\mathbf{N}} + \frac{\tau_0^2}{2}\vec{w}^{(4)}_{\mathbf{N}}, \vec{\varphi}_{N_\theta}\right) = L^\theta_{N_\theta}(\vec{\varphi}_{N_\theta}), \qquad \forall \vec{\varphi}_{N_\theta} \in \vec{V}^\theta_{N_\theta}(\omega), \tag{13}$$

together with the initial conditions

$$\vec{w}_{\mathbf{N}}(0) = \vec{w}_{\mathbf{N}0}, \;\; \vec{w}'_{\mathbf{N}}(0) = \vec{w}_{\mathbf{N}1}, \;\; \vec{\zeta}_{N_\theta}(0) = \vec{\zeta}_{N_\theta 0}, \;\; \vec{\zeta}'_{N_\theta}(0) = \vec{\zeta}_{N_\theta 1}, \;\; \vec{\zeta}''_{N_\theta}(0) = \vec{\zeta}_{N_\theta 2}, \;\; \vec{\zeta}'''_{N_\theta}(0) = \vec{\zeta}_{N_\theta 3} \tag{14}$$

where $\vec{V}_{\mathbf{N}}(\omega) = \{\vec{v}_{\mathbf{N}} = (\overset{r_i}{v}_{\mathbf{N}i}) \in [H^1_{loc}(\omega)]^{N_{1,2,3}}; \|\vec{v}_{\mathbf{N}}\|_{1*} < \infty, tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i}) = 0$ on $\tilde{\gamma}_0, r_i = 0,\dots,N_i, i = 1,2,3\}$, $\vec{H}_{\mathbf{N}}(\omega) = \{\vec{v}_{\mathbf{N}} = (\overset{r_i}{v}_{\mathbf{N}i}) \in [L^2(\omega)]^{N_{1,2,3}}; \;\; \|\vec{v}_{\mathbf{N}}\|^2_{\vec{H}_{\mathbf{N}}(\omega)} = \sum_{i=1}^3 \sum_{r_i=0}^{N_i} \| h^{-1/2} \overset{r_i}{v}_{\mathbf{N}i} \|^2_{L^2(\omega)} < \infty\}$, $\vec{V}^\theta_{N_\theta}(\omega) = \{\vec{\varphi}_{N_\theta} = (\overset{r}{\varphi}_{N_\theta}) \in [H^1_{loc}(\omega)]^{N_\theta+1}$; $\|\vec{\varphi}_{N_\theta}\|_{\theta 1*} < \infty$, $tr_{\tilde{\gamma}}(\overset{r}{\varphi}_{N_\theta}) = 0$ on $\tilde{\gamma}_0$, $r = 0,\dots,N_\theta\}$, $\vec{H}^\theta_{N_\theta}(\omega) = \{\vec{\varphi}_{N_\theta} = (\overset{r}{\varphi}_{N_\theta}) \in [L^2(\omega)]^{N_\theta+1}$; $\|\vec{\varphi}_{N_\theta}\|^2_{\vec{H}^\theta_{N_\theta}(\omega)} = \sum_{r=0}^{N_\theta} \| h^{-1/2} \overset{r}{\varphi}_{N_\theta} \|^2_{L^2(\omega)} < \infty\}$, the bilinear forms $R_{\mathbf{N}}$, $a_{\mathbf{N}}$, $b_{\mathbf{N}N_\theta}$, $R^\theta_{N_\theta}$, $a^\theta_{1N_\theta}$, $a^\theta_{2N_\theta}$, $b^\theta_{\mathbf{N}N_\theta}$ are defined as follows $R_{\mathbf{N}}(\vec{\tilde{v}}_{\mathbf{N}}, \vec{\hat{v}}_{\mathbf{N}}) = \rho(\tilde{\mathbf{v}}_{\mathbf{N}}, \hat{\mathbf{v}}_{\mathbf{N}})_{\mathbf{L}^2(\Omega)}$, $a_{\mathbf{N}}(\vec{\bar{v}}_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) = a(\bar{\mathbf{v}}_{\mathbf{N}}, \mathbf{v}_{\mathbf{N}})$, $b_{\mathbf{N}N_\theta}(\vec{\tilde{\varphi}}_{N_\theta}, \vec{v}_{\mathbf{N}}) = b^\theta_{\mathbf{N}N_\theta}(\vec{v}_{\mathbf{N}}, \vec{\tilde{\varphi}}_{N_\theta}) = b(\tilde{\varphi}_{N_\theta}, \mathbf{v}_{\mathbf{N}})$, $R^\theta_{N_\theta}(\vec{\tilde{\varphi}}_{N_\theta}, \vec{\hat{\varphi}}_{N_\theta}) = \chi(\tilde{\varphi}_{N_\theta}, \hat{\varphi}_{N_\theta})_{L^2(\Omega)}$, $a^\theta_{1N_\theta}(\vec{\bar{\varphi}}_{N_\theta}, \vec{\varphi}_{N_\theta}) = a^\theta_1(\bar{\varphi}_{N_\theta}, \varphi_{N_\theta})$, $a^\theta_{2N_\theta}(\vec{\bar{\varphi}}_{N_\theta}, \vec{\varphi}_{N_\theta}) = a^\theta_2(\bar{\varphi}_{N_\theta}, \varphi_{N_\theta})$, for all vector-functions $\vec{v}_{\mathbf{N}}, \vec{\bar{v}}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega)$, $\vec{\tilde{v}}_{\mathbf{N}}, \vec{\hat{v}}_{\mathbf{N}} \in \vec{H}_{\mathbf{N}}(\omega)$, $\vec{\varphi}_{N_\theta}, \vec{\bar{\varphi}}_{N_\theta} \in \vec{V}^\theta_{N_\theta}(\omega)$, $\vec{\tilde{\varphi}}_{N_\theta}, \vec{\hat{\varphi}}_{N_\theta} \in \vec{H}^\theta_{N_\theta}(\omega)$, corresponding to $\mathbf{v_N}, \overline{\mathbf{v}}_\mathbf{N} \in \mathbf{V_N}(\Omega)$, $\tilde{\mathbf{v}}_\mathbf{N}, \hat{\mathbf{v}}_\mathbf{N} \in \mathbf{H_N}(\Omega)$, $\varphi_{N_\theta}, \overline{\varphi}_{N_\theta} \in V^\theta_{N_\theta}(\Omega)$, $\tilde{\varphi}_{N_\theta}, \hat{\varphi}_{N_\theta} \in H^\theta_{N_\theta}(\Omega)$, respectively. The linear forms $L_\mathbf{N}$ and $L^\theta_{N_\theta}$ are defined by the right-hand sides of the equations (7), (8) and are given by

$$L_\mathbf{N}(\vec{v}_\mathbf{N}) = \sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\left(r_i+\frac{1}{2}\right)\left[\int_\omega \frac{1}{h}\overset{r_i}{v}_{\mathbf{N}i}\left(\overset{r_i}{f}_{\mathbf{N}i}+g^+_{\mathbf{N}i}\lambda_+ + g^-_{\mathbf{N}i}\lambda_-(-1)^{r_i}\right)d\omega + \int_{\gamma_1}\frac{1}{h}tr_{\gamma_1}(\overset{r_i}{v}_{\mathbf{N}i})\overset{r_i}{g}_{\mathbf{N}i}\,d\gamma_1\right],$$

$$L^\theta_{N_\theta}(\vec{\varphi}_{N_\theta}) = \sum_{r=0}^{N_\theta}\left(r+\frac{1}{2}\right)\left[\int_\omega \frac{1}{h}\overset{r}{\varphi}_{N_\theta}\left(\overset{r}{f}{}^\theta_{N_\theta}-g^{\theta+}_{N_\theta}\lambda_+ - g^{\theta-}_{N_\theta}\lambda_-(-1)^{r}\right)d\omega - \int_{\gamma_1}\frac{1}{h}tr_{\gamma_1}(\overset{r}{\varphi}_{N_\theta})\overset{r}{g}{}^\theta_{N_\theta}\,d\gamma_1\right],$$

where $\gamma_1 = \tilde{\gamma}\setminus\tilde{\gamma}_0$, $\lambda_\pm = \sqrt{1+(\partial_1 h^\pm)^2+(\partial_2 h^\pm)^2}$, $\overset{r}{v} = \int_{h^-}^{h^+} vP_r(z)dx_3$, for all functions $v \in L^2(\Omega)$, $r \in \mathbf{N}\cup\{0\}$, $\overset{+}{g}_{\mathbf{N}i}$,, $g^{\theta+}_{N_\theta}$ and $\overset{-}{g}_{\mathbf{N}i}$,, $g^{\theta-}_{N_\theta}$ are restrictions of

$$g_{\mathbf{N}i}(t) = g_i(t) + \sum_{k=0}^{3}\frac{t^k}{k!}\left(\sum_{j=1}^{3}tr_{\Gamma_1}\left(\lambda\sum_{p=1}^{3}e_{pp}(\mathbf{w}_{\mathbf{N}k})\delta_{ij} + 2\mu e_{ij}(\mathbf{w}_{\mathbf{N}k}) + \eta\zeta_{N_\theta k}\delta_{ij}\right)\nu_j - g_i^{(k)}(0)\right),\ i=1,2,3,$$

$$w_{\mathbf{N}\,\alpha+2\,i} = \frac{1}{\rho}\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\lambda\sum_{p=1}^{3}e_{pp}(\mathbf{w}_{\mathbf{N}\alpha})\delta_{ij} + 2\mu e_{ij}(\mathbf{w}_{\mathbf{N}\alpha}) + \eta\zeta_{N_\theta\alpha}\delta_{ij}\right) + f^{(\alpha)}_{\mathbf{N}i}(0)\right),\ \alpha=0,1,\ i=1,2,3,$$

$$g^\theta_{N_\theta}(t) = g^\theta(t) - \sum_{j=1}^{3}\kappa\, tr_{\Gamma_1}\left(\frac{\partial\zeta_{N_\theta 1}}{\partial x_j} + \tau_1\frac{\partial\zeta_{N_\theta 2}}{\partial x_j}\right)\nu_j - \sum_{j=1}^{3}\overline{\kappa}\, tr_{\Gamma_1}\left(\frac{\partial\zeta_{N_\theta 0}}{\partial x_j} + \tau_2\frac{\partial\zeta_{N_\theta 1}}{\partial x_j}\right)\nu_j - g^\theta(0),$$

on the upper Γ^+ and the lower Γ^- surfaces of the plate, respectively, $\mathbf{v} = (\nu_i)_{i=1}^3$ is the unit outward normal vector to Γ, $\mathbf{w}_{\mathbf{N}0} \in \mathbf{V}^5_\mathbf{N}(\Omega)$, $\mathbf{w}_{\mathbf{N}1} \in \mathbf{V}^4_\mathbf{N}(\Omega)$, $\zeta_{N_\theta 0} \in V^{\theta,4}_{N_\theta}(\Omega)$, $\zeta_{N_\theta 1} \in V^{\theta,3}_{N_\theta}(\Omega)$, $\zeta_{N_\theta 2} \in V^{\theta,2}_{N_\theta}(\Omega)$ are restored from the initial conditions $\vec{w}_{\mathbf{N}0} \in \vec{V}^5_\mathbf{N}(\omega)$, $\vec{w}_{\mathbf{N}1} \in \vec{V}^4_\mathbf{N}(\omega)$, $\vec{\zeta}_{N_\theta 0} \in \vec{V}^{\theta,4}_{N_\theta}(\omega)$, $\vec{\zeta}_{N_\theta 1} \in \vec{V}^{\theta,3}_{N_\theta}(\omega)$, $\vec{\zeta}_{N_\theta 2} \in \vec{V}^{\theta,2}_{N_\theta}(\omega)$ of the two-dimensional problem, where $\vec{V}^k_\mathbf{N}(\omega) = \vec{H}^k_\mathbf{N}(\omega)\cap\vec{V}_\mathbf{N}(\omega)$, $\vec{H}^k_\mathbf{N}(\omega) = \{\vec{v}_\mathbf{N} = (\overset{r_i}{v}_{\mathbf{N}i}) \in [H^k_{loc}(\omega)]^{N_{1,2,3}}$; $\|\vec{v}_\mathbf{N}\|_{k*} < \infty\}$, $k=2,3,4,5$, $\vec{V}^{\theta,k_\theta}_{N_\theta}(\omega) = \{\vec{\varphi}_{N_\theta} = (\overset{r}{\varphi}_{N_\theta}) \in [H^{k_\theta}_{loc}(\omega)]^{N_\theta+1} \cap \vec{V}^\theta_{N_\theta}(\omega); \|\vec{\varphi}_{N_\theta}\|_{\theta k_\theta *} < \infty\}$, $k_\theta = 2,3,4$.

The following existence and uniqueness theorem is proved for the two-dimensional initial-boundary value problems (12)-(14) of the constructed hierarchy.

Theorem 2. Suppose that the two-dimensional domain ω and functions h^+ and h^- are such that $\Omega \subset \mathbf{R}^3$ is a Lipschitz domain, $\rho > 0$, $\mu > 0$, $3\lambda + 2\mu > 0$, $\chi > 0$, $\kappa > 0$, $\overline{\kappa} > 0$, $\tau_0 > 0$, $\tau_1 > 0$, $\tau_2 > 0$. If the given functions satisfy the following conditions

$$\vec{f}_\mathbf{N} = (\overset{r_i}{f}_{\mathbf{N}i}) \in C([0,T];\vec{H}^3_\mathbf{N}(\omega)),\ (\vec{f}_\mathbf{N})' \in C([0,T];\vec{H}^2_\mathbf{N}(\omega)),\ (\vec{f}_\mathbf{N})'' \in C([0,T];\vec{H}^1_\mathbf{N}(\omega)),$$

$$h^{-1/2}(\overset{r_i}{f}_{\mathbf{N}i})''', h^{-1/2}(\overset{r_i}{f}_{\mathbf{N}i})^{(4)} \in L^2(0,T;L^2(\omega)),\ h^{-1/2}\overset{r}{f}{}^\theta_{N_\theta}, h^{-1/2}\frac{d\overset{r}{f}{}^\theta_{N_\theta}}{dt} \in L^2(0,T;L^2(\omega)),$$

$$\lambda_\pm^{3/4}\frac{d^\alpha g^\pm_{\mathbf{N}i}}{dt^\alpha} \in L^2(0,T;L^{4/3}(\omega)),\ h^{-1/4}\frac{d^\alpha \overset{r_i}{g}_{\mathbf{N}i}}{dt^\alpha} \in L^2(0,T;L^{4/3}(\gamma_1)),\ \alpha = 0,1,\ldots,5,\ r_i = 0,\ldots,N_i, i=1,2,3,$$

$$\lambda_\pm^{3/4}\frac{d^\beta g^{\theta\pm}_{N_\theta}}{dt^\beta} \in L^2(0,T;L^{4/3}(\omega)),\ h^{-1/4}\frac{d^\beta \overset{r}{g}{}^\theta_{N_\theta}}{dt^\beta} \in L^2(0,T;L^{4/3}(\gamma_1)),\ \beta = 0,1,2,\ r = 0,\ldots,N_\theta,$$

and $\vec{w}_{\mathbf{N}0} \in \vec{V}^5_\mathbf{N}(\omega)$, $\vec{w}_{\mathbf{N}1} \in \vec{V}^4_\mathbf{N}(\omega)$, $\vec{\zeta}_{N_\theta 0} \in \vec{V}^{\theta,4}_{N_\theta}(\omega)$, $\vec{\zeta}_{N_\theta 1} \in \vec{V}^{\theta,3}_{N_\theta}(\omega)$, $\vec{\zeta}_{N_\theta 2} \in \vec{V}^{\theta,2}_{N_\theta}(\omega)$, $\vec{\zeta}_{N_\theta 3} \in \vec{V}^{\theta}_{N_\theta}(\omega)$, then the two-dimensional initial-boundary value problem (12)-(14) possesses a unique solution.

So, we have constructed an algorithm of approximation of the three-dimensional model with three phase-lags for thermoelastic plates with variable thickness by two-dimensional initial-boundary value problems. In the following theorem we give the results on the convergence of the algorithm, where we assume, that the functions h^+ and h^-, which define the upper and the lower surfaces of the plate, and their derivatives up the fourth order are Lipschtiz continuous on the domain $\omega \subset \mathbf{R}^2$, i.e. $h^\pm \in C^{4,1}(\overline{\omega})$, and we use the following anisotropic weighted Sobolev spaces

$$H^{0,0,s}_{h^\pm}(\Omega) = \{v; h^k \partial_3^k v \in L^2(\Omega), 0 \le k \le s\}, \quad s \in \mathbf{N},$$

$$H^{m,m,s}_{h^\pm}(\Omega) = \{v;\ h^{k-m}\partial_3^{k-m}\partial_{i_1}^{r_1}\partial_{i_2}^{r_2}...\partial_{i_m}^{r_m} v \in L^2(\Omega),\ m \le k \le s,\ r_1, r_2, ..., r_m = 0,1,\ i_1, i_2, ..., i_m = 1,2,3\}, s \ge m, m \in \mathbf{N},$$

which are Hilbert spaces with respect to the corresponding norms.

Theorem 3. If $\Omega \subset \mathbf{R}^3$ is a bounded Lipschitz domain, $\rho > 0$, $\mu > 0$, $3\lambda + 2\mu > 0$, $\chi > 0$, $\kappa > 0$, $\overline{\kappa} > 0$, $\tau_0 > 0$, $\tau_1 > 0$, $\tau_2 > 0$, $\mathbf{f} \in C([0,T]; \mathbf{H}^3(\Omega))$, $\mathbf{f}' \in C([0,T]; \mathbf{H}^2(\Omega))$, $\mathbf{f}'' \in C([0,T]; \mathbf{H}^1(\Omega))$, $\mathbf{f}''', \mathbf{f}^{(4)} \in L^2(0,T; \mathbf{L}^2(\Omega))$, $\mathbf{g}, \mathbf{g}', \mathbf{g}'', \mathbf{g}''', \mathbf{g}^{(4)}, \mathbf{g}^{(5)} \in L^2(0,T; \mathbf{L}^{4/3}(\Gamma_1))$, $f^\theta, f^{\theta'} \in L^2(0,T; L^2(\Omega))$, $g^\theta, g^{\theta'}$, $g^{\theta''} \in L^2(0,T; L^{4/3}(\Gamma_1))$, $\mathbf{u}_0 \in \mathbf{H}^5(\Omega) \cap \mathbf{V}(\Omega)$, $\mathbf{u}_1 \in \mathbf{H}^4(\Omega) \cap \mathbf{V}(\Omega)$, $\theta_0 \in H^4(\Omega) \cap V^\theta(\Omega)$, $\theta_1 \in H^3(\Omega) \cap V^\theta(\Omega)$, $\theta_2 \in H^2(\Omega) \cap V^\theta(\Omega)$, $\theta_3 \in V^\theta(\Omega)$ satisfy the compatibility conditions (10), (11) and the functions $\mathbf{w}_{\mathbf{N}0} \in \mathbf{V}^5_{\mathbf{N}}(\Omega)$, $\mathbf{w}_{\mathbf{N}1} \in \mathbf{V}^4_{\mathbf{N}}(\Omega)$, $\zeta_{N_\theta 0} \in V^{\theta,4}_{N_\theta}(\Omega)$, $\zeta_{N_\theta 1} \in V^{\theta,3}_{N_\theta}(\Omega)$, $\zeta_{N_\theta 2} \in V^{\theta,2}_{N_\theta}(\Omega)$, $\zeta_{N_\theta 3} \in V^{\theta}_{N_\theta}(\Omega)$, corresponding to the initial conditions $\vec{w}_{\mathbf{N}0} \in \vec{V}^5_{\mathbf{N}}(\omega)$, $\vec{w}_{\mathbf{N}1} \in \vec{V}^4_{\mathbf{N}}(\omega)$, $\vec{\zeta}_{N_\theta 0} \in \vec{V}^{\theta,4}_{N_\theta}(\omega)$, $\vec{\zeta}_{N_\theta 1} \in \vec{V}^{\theta,3}_{N_\theta}(\omega)$, $\vec{\zeta}_{N_\theta 2} \in \vec{V}^{\theta,2}_{N_\theta}(\omega)$, $\vec{\zeta}_{N_\theta 3} \in \vec{V}^{\theta}_{N_\theta}(\omega)$ of two dimensional problems, tend to $\mathbf{u}_0$, $\mathbf{u}_1$, θ_0, θ_1, θ_2 and θ_3 in the spaces $\mathbf{H}^4(\Omega)$, $\mathbf{H}^4(\Omega)$, $H^3(\Omega)$, $H^3(\Omega)$, $H^2(\Omega)$ and $H^1(\Omega)$, respectively, and the vector-functions of three-space variables $\mathbf{f}_{\mathbf{N}} \in C([0,T]; \mathbf{H}^3_{\mathbf{N}}(\Omega))$ and $f^\theta_{N_\theta} \in L^2(0,T; H^\theta_{N_\theta}(\Omega))$ corresponding to the vector-functions $\vec{f}_{\mathbf{N}} = (\overset{r_i}{f}_{\mathbf{N}i}) \in C([0,T]; \vec{H}^3_{\mathbf{N}}(\omega))$ and $\vec{f}^\theta_{N_\theta} = (\overset{r}{f}{}^\theta_{N_\theta}) \in L^2(0,T; \vec{H}^\theta_{N_\theta}(\omega))$ are such that $\mathbf{f}_{\mathbf{N}}$, $\mathbf{f}'_{\mathbf{N}}$, $\mathbf{f}''_{\mathbf{N}}$, $\mathbf{f}'''_{\mathbf{N}}$ tend to $\mathbf{f}$, $\mathbf{f}'$, $\mathbf{f}''$, $\mathbf{f}'''$ in $L^2(0,T; \mathbf{L}^2(\Omega))$, $\mathbf{f}_{\mathbf{N}}(0)$ tends to $\mathbf{f}(0)$ in $\mathbf{H}^2(\Omega)$, $\mathbf{f}'_{\mathbf{N}}(0)$ tends to $\mathbf{f}'(0)$ in $\mathbf{H}^2(\Omega)$, and $f^\theta_{N_\theta}$ converges to f^θ in the space $L^2(0,T; L^2(\Omega))$, as $N_{\min} = \min_{1 \le i \le 3}\{N_i, N_\theta\} \to \infty$, then the two-dimensional problem (12)-(14) posseses a unique solution and the sequences of vector-functions $\mathbf{w}_{\mathbf{N}}(t)$ and functions $\zeta_{N_\theta}(t)$, restored from the solutions $\vec{w}_{\mathbf{N}}(t)$ and $\vec{\zeta}_{N_\theta}(t)$ of the problem (12)-(14), tend to the solutions $\mathbf{u}(t)$ and $\theta(t)$ of the original three-dimensional problem (7)-(9),

$$\begin{aligned}
&\frac{d^\alpha \mathbf{w}_{\mathbf{N}}}{dt^\alpha}(t) \to \frac{d^\alpha \mathbf{u}}{dt^\alpha}(t) && \text{in } \mathbf{H}^1(\Omega), \quad \forall t \in [0,T], \alpha = 0,1,2, \\
&\mathbf{w}'''_{\mathbf{N}}(t) \to \mathbf{u}'''(t) && \text{in } \mathbf{L}^2(\Omega), \quad \forall t \in [0,T], && \text{as } N_{\min} \to \infty, \\
&\mathbf{w}'''_{\mathbf{N}} \to \mathbf{u}''' && \text{in } L^2(0,T; \mathbf{H}^1(\Omega)), \\
&\mathbf{w}^{(4)}_{\mathbf{N}} \to \mathbf{u}^{(4)} && \text{in } L^2(0,T; \mathbf{L}^2(\Omega)),
\end{aligned}$$

$$\begin{aligned}
&\frac{d^\beta \zeta_{N_\theta}}{dt^\beta}(t) \to \frac{d^\beta \theta}{dt^\beta}(t) && \text{in } H^1(\Omega), \quad \forall t \in [0,T], \beta = 0,1, \\
&\zeta''_{N_\theta}(t) \to \theta''(t) && \text{in } L^2(\Omega), \quad \forall t \in [0,T], && \text{as } N_{\min} \to \infty. \\
&\zeta''_{N_\theta} \to \theta'' && \text{in } L^2(0,T; H^1(\Omega)), \\
&\zeta'''_{N_\theta} \to \theta''' && \text{in } L^2(0,T; L^2(\Omega)),
\end{aligned}$$

In addition, if $d^r\mathbf{u}/dt^r \in L^2(0,T;(H^{1,1,s_r}_{h^\pm}(\Omega))^3)$, $s_r \ge 5$, $s_r \in \mathbf{N}$, $r=1,2,3,4$, $\mathbf{u}^{(5)} \in L^2(0,T;(H^{0,0,s_5}_{h^\pm}(\Omega))^3)$, $s_5 \ge 4$, $s_5 \in \mathbf{N}$, $d^{\tilde r}\theta/dt^{\tilde r} \in L^2(0,T;H^{1,1,s^\theta_{\tilde r}}_{h^\pm}(\Omega))$, $s^\theta_{\tilde r} \ge 4$, $s^\theta_{\tilde r} \in \mathbf{N}$, $\tilde r = 0,1,2,3$, $\theta^{(4)} \in L^2(0,T;\ H^{0,0,s^\theta_4}_{h^\pm}(\Omega))$, $s^\theta_4 \ge 3$, $s^\theta_4 \in \mathbf{N}$, and $\mathbf{u}_0 \in (H^{4,4,\tilde s_0}_{h^\pm}(\Omega))^3 \cap (H^{5,5,9}_{h^\pm}(\Omega))^3$, $\tilde s_0 \ge 8$, $\mathbf{u}_1 \in (H^{3,3,\tilde s_1}_{h^\pm}(\Omega))^3 \cap (H^{4,4,8}_{h^\pm}(\Omega))^3$, $\tilde s_1 \ge 7$, $\theta_0 \in H^{3,3,\tilde s^\theta_0}_{h^\pm}(\Omega) \cap H^{4,4,7}_{h^\pm}(\Omega)$, $\tilde s^\theta_0 \ge 6$, $\theta_1 \in H^{2,2,\tilde s^\theta_1}_{h^\pm}(\Omega) \cap H^{3,3,6}_{h^\pm}(\Omega)$, $\tilde s^\theta_1 \ge 5$, $\theta_2 \in H^{1,1,\tilde s^\theta_2}_{h^\pm}(\Omega) \cap H^{2,2,5}_{h^\pm}(\Omega)$, $\tilde s^\theta_2 \ge 4$, $\tilde s_0, \tilde s_1, \tilde s^\theta_0, \tilde s^\theta_1, \tilde s^\theta_2 \in \mathbf{N}$, $\theta_3 \in H^{1,1,4}_{h^\pm}(\Omega)$, $d^p\mathbf{f}/dt^p \in L^2(0,T;(H^{0,0,\hat s_p}_{h^\pm}(\Omega))^3)$, $p=1,2,3$, $\hat s_1,\hat s_2,\hat s_3 \ge 2$, $\hat s_1,\hat s_2,\hat s_3 \in \mathbf{N}$, $\mathbf{f}(0) \in (H^{2,2,\overline{s}_0}_{h^\pm}(\Omega))^3$, $\overline{s}_0 \ge 4$, $\mathbf{f}'(0) \in (H^{1,1,\overline{s}_1}_{h^\pm}(\Omega))^3$, $\overline{s}_1 \ge 3$, $\mathbf{f}''(0) \in (H^{0,0,\overline{s}_2}_{h^\pm}(\Omega))^3$, $\overline{s}_2 \ge 2$, $\overline{s}_0, \overline{s}_1, \overline{s}_2 \in \mathbf{N}$, then for appropriate initial conditions $\vec w_{\mathbf{N}0}$, $\vec w_{\mathbf{N}1}$, $\vec\zeta_{N_\theta 0}$, $\vec\zeta_{N_\theta 1}$, $\vec\zeta_{N_\theta 2}$, $\vec\zeta_{N_\theta 3}$ and $\vec f_{\mathbf{N}}$ the following estimate is valid

$$\|\mathbf{u}-\mathbf{w_N}\|_{C([0,T];\mathbf{H}^1(\Omega))} + \|\mathbf{u}'-\mathbf{w}'_\mathbf{N}\|_{C([0,T];\mathbf{H}^1(\Omega))} + \|\mathbf{u}''-\mathbf{w}''_\mathbf{N}\|_{C([0,T];\mathbf{H}^1(\Omega))} + \|\mathbf{u}'''-\mathbf{w}'''_\mathbf{N}\|_{C([0,T];\mathbf{L}^2(\Omega))} +$$
$$+\|\mathbf{u}'''-\mathbf{w}'''_\mathbf{N}\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \|\mathbf{u}^{(4)}-\mathbf{w}^{(4)}_\mathbf{N}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\theta-\zeta_{N_\theta}\|_{C([0,T];H^1(\Omega))} + \|\theta'-\zeta'_{N_\theta}\|_{C([0,T];H^1(\Omega))} +$$
$$+\|\theta''-\zeta''_{N_\theta}\|_{C([0,T];L^2(\Omega))} + \|\theta''-\zeta''_{N_\theta}\|_{L^2(0,T;H^1(\Omega))} + \|\theta'''-\zeta'''_{N_\theta}\|_{L^2(0,T;L^2(\Omega))} \le \frac{1}{(N_{\min})^s} o(T,\Omega,\tilde\Gamma_0,h^\pm,\mathbf{N},N_\theta),$$

where $s = \min\{\min_{1\le i\le 4}\{s_i - 1\}, s_5, s^\theta_0, \min_{1\le j\le 3}\{s^\theta_j - 1, \hat s_j\}, s^\theta_4, \tilde s_0 - 4, \tilde s_1 - 3, \tilde s^\theta_0 - 3, \tilde s^\theta_1 - 2, \tilde s^\theta_2 - 1, \overline{s}_0 - 2, \overline{s}_1 - 1, \overline{s}_2\}$ and $o(T,\Omega,\tilde\Gamma_0,h^\pm,\mathbf{N},N_\theta) \to 0,$ as $N_{\min} \to \infty$.

მათემატიკა

თერმოდრეკადი ფირფიტების სამფაზიანი დაგვიანებით არაკლასიკური სამგანზომილებიანი მოდელის ორგანზომილებიანი ამოცანებით აპროქსიმაციის შესახებ

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(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

ნაშრომში ვარიაციული მიდგომის გამოყენებით შესწავლილია სამფაზიანი დაგვიანებით არაკლასიკური დინამიკური მოდელი თერმოდრეკადი ფირფიტისათვის ცვლადი სისქით, რომელიც შეიძლება ნულის ტოლი იყოს გვერდითი საზღვრის ნაწილზე. აგებულია სამგანზომილებიანი მოდელის ორგანზომილებიანი ამოცანებით აპროქსიმაციის ალგორითმი, როცა ფირფიტის ზედა და ქვედა პირით ზედაპირებზე მოცემულია ზედაპირული ძალის და საზღვრის გარე ნორმალის გასწვრივ სითბოს ნაკადის სიმკვრივეები. აგებული ორგანზომილებიანი საწყის-სასაზღვრო ამოცანები შესწავლილია სათანადო ფუნქციათა სივრცეებში, დამტკიცებულია ორგანზომილებიანი ამოცანების ამონახსნებიდან აღდგენილი სამი სივრცითი ცვლადის ვექტორ-ფუნქციების მიმდევრობის კრებადობა შესაბამის სივრცეებში საწყისი სამგანზომილებიანი ამოცანის ამონახსნისაკენ და დამატებით პირობებში შეფასებულია კრებადობის რიგი.

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