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Mathematics

On the Topological Degree of the Gradient of Homogeneous Gaussian Random Polynomial

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(Presented by Academy Member Elizbar Nadaraya)

A class of Gaussian random polynomials is introduced for which it appears possible to compute the expected value of the topological degree of gradient. For an odd number of variables, an explicit formula for the expected value of the topological degree of gradient is given in terms of the so-called parameter of the considered random polynomial. For an even number of variables, the asymptotic of the expected value of absolute degree is given. The proofs make essential use of the recent results of I. Ibragimov and S. Podkorytov on the expected value of the Euler characteristic of random hypersurface. *© 2024 Bull. Georg. Natl. Acad. Sci.*

normal random variable, random polynomial, gradient, topological degree, Euler characteristic

We consider random polynomials with coefficients which are normal random variables. Such setting is quite natural and closely related to interesting problems of mathematics and physics [1].

For polynomials of one variable, a natural problem is to compute the expected number of real roots. In certain cases, this problem was solved in the seminal paper of M. Kac [1]. Nowadays there exists vast literature devoted to random polynomials of one variable and the expected number of real roots is known for many classes of univariate random polynomials.

Much less is known about random polynomials of several variables. The most natural setting is to consider n random polynomials of n variables with fixed algebraic multi-degree $m=(m_1,...,m_n)$ and compute the expected number of their common real roots. In other words, this is a question about the average number of real roots of the corresponding $(n\times n)$–system of random polynomial equations. M. Shub and S. Smale [2] succeeded in computing this invariant for certain special distributions of coefficients, which may be considered as a wide generalization of the mentioned classical results by M. Kac. These and other recent developments in the spirit of [1] are summarized in [3].

Despite this progress, many natural topics concerned with multivariable random polynomials remain uninvestigated. We discuss here one of such topics where some progress can be achieved using the techniques presented in [4, 5]. The problem we consider below, is to compute the expected value of the

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topological degree of random polynomial endomorphism defined by Teimuraz Aliashvilli normal random polynomials of n variables. The polynomials are assumed to be homogeneous of fixed multi-degrees as above. Taking into account results of [5], it becomes clear that solution of this problem would enable one to obtain many results about the topology of random polynomials.

Consider a Gaussian random polynomial on $\mathbb{R}^{n+1}$ of algebraic degree N such that each of its homogeneous components of degree $m \le N$ has rotation invariant Gaussian distribution. In such situation we speak of a rotation invariant Gaussian random polynomial of algebraic degree N. This is precisely the type of random polynomial we wish to consider. The main reason for such choice is that, in this case, there exist comprehensive results about the expected number of real roots [2,3] and the mean value of Euler characteristic of the corresponding random hypersurface [4] one of which is used in the sequel.

We give now a rigorous description of the problems we are interested in. Suppose we are given a Gaussian random polynomial P in s variables. Then we may consider the canonical random polynomial mapping $F: \mathbb{R}^s \to \mathbb{R}^t$ with the components given by t independent random polynomial with the same distribution as P. For $t = s$ we get a random polynomial endomorphism associated with the given Gaussian distribution. Of course, one need not require that the components of F are independent and have the same distribution, and in this way, we obtain the notion of a general random polynomial mapping.

A whole series of nontrivial problems can be formulated in terms of computing the expected values (mean values) of various topological invariants of the fibers of such random polynomial mappings. For example, for $t = 1$ the average Euler characteristic of fibers in certain cases was found in [4]. Most of such problems make sense for arbitrary $t \le s$ but, for brevity and simplicity, here we only discuss random polynomial endomorphisms.

Notice that such a random polynomial endomorphism F is almost certainly (a.c.) proper, so its fibers are a.c. finite and one can wonder what the expected number of real roots of is the corresponding random polynomial $(s \times s)$ system. Analogously, the mapping degree $DegF$ is a.c. well-defined and one may wish to compute the mathematical expectation $E(DegF)$ of this random variable and /or of its absolute value $E(|DegF|)$. We call these expectations the expected mapping degree of F and expected absolute degree of F, respectively.

For a canonical random endomorphism associated with a Gaussian polynomial as above, the expected number of real roots was computed in [2,3]. However, it remained unclear how to find an explicit formula for its expected mapping degree or expected absolute degree. Below we estimate these invariants in a different setting. Namely, we consider a random endomorphism $F = P'$ defined as the gradient mapping of a given random polynomial P, i.e. the components of P' are given by the partial derivatives of P. Notice that in this case the components of F are neither independent nor identically distributed so this is an example of a general random endomorphism. The expectation $E(DegP')$ will be called the expected gradient degree of P.

To compute this invariant, we further assume that P is a Gaussian polynomial as in the above example. First, we formulate the main result in the case when the source is even-dimensional (notice that the source dimension in our example is denoted by $n+1$).

Thus, we suppose that n is odd and P is a rotation invariant Gaussian random polynomial of algebraic degree m. Introduce the number $M_n(m)$ by equality:

$$M_n(m) = \frac{I_n(\sqrt{m})}{I_n(1)},$$

where

$$I_n(s) = \int_0^s (1-t^2)^{\frac{n-1}{2}} dt, \quad s \in \mathbb{R}_+ .$$

Theorem 1. For an odd $n \geq 1$, the expectation of gradient degree $E(DegP')$ of rotation invariant Gaussian random polynomial P of algebraic degree m on $\mathbb{R}^{n+1}$ is equal to $1 - M_n(m)$.

Proof. Consider the random hypersurface V_P in $\mathbb{R}P^n$ defined by the homogeneous equation $\{Q = 0\}$, where $Q = P^*$ is the homogeneous form of highest degree in F (thus the real dimension of V_P is $n-1$). As was proved in [3], in this situation hypersurface V_Q is almost certainly smooth and its Euler characteristic $\chi(V_Q)$ is well-defined. We will deduce the result from the formula for the average $E\left(\chi\left(V_Q\right)\right)$ obtained in [4]. To this end, we use the formula expressing the Euler characteristic of hypersurface through the gradient degree of its defining polynomial which was established in [5].

Namely, denoting by Y the intersection of hypersurface $\{Q = 0\}$ with the unit sphere S^n and using the main theorem of [5], we get:

$$\chi(Y) = 2\left(1 - DegQ'\right).$$

Taking into account that Y is the double of V_Q, we conclude that:

$$\chi\left(V_Q\right) = 1 - DegQ'.$$

Notice now that almost certainly $DegQ' = DegP'$. Indeed, it is well known (see, e.g., [6]) that this equality takes place as long as the components of homogeneous endomorphism $Q' = \left(P^*\right)'$ have no common nontrivial zeros and the latter circumstance obviously happens with probability one. Thus, taking the expectations of both sides of the last equality we get

$$E\left(\chi(V_Q)\right) = 1 - E(DegQ') = 1 - E(DegP') .$$

It remains to add that, according to the main result of [4], in our situation the expectation $E\left(\chi\left(V_Q\right)\right)$ is equal to $M_n(m)$ introduced above since the parameter of our random polynomial is equal to m. This completes the proof.

Remark 1. From this proof one can actually conclude that it is sufficient to require only that the highest order form $P^* \in H^m(\mathbb{R}^{n+1})$ is a rotation invariant random homogeneous polynomial and the terms of lower degree can have arbitrary central Gaussian distributions. The answer can again be expressed in terms of the parameter of P^*.

Since the Euler characteristic vanishes for all compact odd-dimensional smooth manifolds, the above argument does not work for even n. So, to obtain a more general result we have to consider the expected absolute degree $E(|DegP'|)$. Using the additivity of the Euler characteristic one can estimate it in all dimensions and find its asymptotic when m tends to infinity.

Theorem 2. The expectation of absolute gradient degree $E(|DegP'|)$ of rotation invariant Gaussian random polynomial of algebraic degree m on $\mathbb{R}^{n+1}$ is asymptotically equivalent to $n^{-1}m^{\frac{n}{2}}$ when m tends to infinity.

Notice that for $n=1$, one gets $M_1(r)=\sqrt{r}$ so the value of $\chi(V_f)$ in this case has indeed the asymptotic indicated above. It is also possible to use the above argument for deriving an exact formula for the average of the absolute gradient degree of a random affine endomorphism which is given as gradient mapping of a convenient random polynomial in arbitrary dimension.

As was already mentioned, the problem of computing the expected mapping degree for an arbitrary Gaussian random endomorphism is open and seems very difficult. Another open problem is to compute the expected Euler characteristic of projectivized fibers of a homogeneous random pro-map. This problem is open even for quadratic mappings. Obviously, similar problems can be formulated for many other classes of random polynomials but, as far as we know, there are no results for non-Gaussian random polynomials.

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გაუსის ერთგვაროვანი შემთხვევითი მრავალწევრის გრადიენტის ტოპოლოგიური ხარისხის შესახებ

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ილიას სახელმწიფო უნივერსიტეტი, ბიზნესის, ტექნოლოგიისა და განათლების ფაკულტეტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

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REFERENCES

1. Kac M. (1943) On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.* **49:** 314-320.
2. Shub M., Smale S. (1993) Complexity of Bezout's theorem II. *Volumes and probabilities. Progr. Math,* **109**: 267-285.
3. Edelman A., Kostlan E. (1995) How many roots of a real polynomial are real? *Bull. Amer. Math. Soc.***106**: 19-38.
4. Podkorytov S. (1999) Srednee znachenie Eilerovoi kharakteristiki sluchainoi algebraicheskoi giperpoverkhnosti. *Algebra i Analiz* **11**: 185-193 (in Russian).
5. Khimshiashvili G. (1977) On the local degree of a smooth mapping*. Bull. Acad. Sci. Georgian SSR.* **85**: 31-34.

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