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Physics

Further Paradoxes in Quantum Mechanics

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In this paper, some new paradoxes are found following from the tracelessness of commutators of physical operators in quantum mechanics. Due to the unbounded nature of most operators in quantum mechanics, the trace is not defined correctly for all operators, leading to some contradictory results. This concerns operators that can be represented as a commutator of two other operators. It is shown that the sum of all eigenvalues of such operators, being Hermitian or self-adjoint, is zero. The relevant examples and consequenses are also considered.

phytoremediation, urbanization, toxicants, heavy metals

A major part of operators in quantum mechanics is unbounded operators in the Hilbert space [1]. For them, the Hilbert spaces are of infinite dimensions, therefore some mathematical problems appear by which physical observables' properties sharply differ from those in finite-dimensional spaces. Recently, many papers and review articles have been devoted to the application of relevant subjects of functional analysis (mainly the theory of the linear operators) in quantum mechanics.

In [2], a series of simple examples illustrate how a lack of mathematical concern can readily lead to surprising mathematical contradictions in wave mechanics (see also [3, 4]).

One of the first paradoxes belongs to the fundamental commutation relation between canonical conjugate variables – coordinate $\left(\hat{Q}\right)$ and linear momentum $\left(\hat{P}\right)$ operators

$$\left[\hat{Q},\hat{P}\right]=i\hbar\hat{I} \tag{1}$$

from which a basic quantum mechanical relation – Heisenberg's uncertainty relation $\Delta x\cdot\Delta p_x\geq\frac{\hbar}{2}$ has been often derived [5]. But if we take the trace of (1), one finds a contradictory result: for the left-hand side, $Tr\left[\hat{Q},\hat{P}\right]=0$, whereas for the right-hand side, $Tr\left(i\hbar\hat{I}\right)\neq 0$.

To resolve this difficulty, one has to admit that Heisenberg's relation cannot be realized in finite-dimensional Hilbert spaces. Quantum mechanics has to be formulated in infinite-dimensional Hilbert space [1] (more probably, in rigged Hilbert space [6]), where the trace is no longer a well-defined operation for all operators and, therefore, one can no longer deduce contradiction from the commutation relation in the indicated manner. Accordingly, at least, one of the operators $\hat{Q}$ or $\hat{P}$, satisfying the commutation relation (1), must be unbounded and therefore, this relation cannot be discussed without considering the domains of definition of these operators.

However, the difficulties of another variety appear for traces in general. Indeed, if $\hat{A}$ and $\hat{B}$ are Hermitian (more precisely, self-adjoint) operators such that

$$\left[\hat{A},\hat{B}\right]=i\hat{C}\,, \qquad (2)$$

where $\hat{C}$ is another Hermitian operator, then, by definition $Tr\left(\hat{A}\hat{B}\right)=Tr\left(\hat{B}\hat{A}\right)$ it always follows

$$Tr\left[\hat{A},\hat{B}\right]=0 \qquad (3)$$

or

$$Tr\left(\hat{C}\right)=0\,, \qquad (4)$$

i.e. the trace of arbitrary operator, if it is expressible by the commutator of another two operators is always zero. This property may impose some limitations on operator $\hat{C}$.

For further elucidation, let us consider several examples. In Eq. (3), $Tr\left(\hat{C}\right)$ means the following: we must average $\hat{C}$ by some function from the complete set of eigenfunctions and sum over the entire spectrum. As these functions at the same time are eigenfunctions of the operator under consideration $\hat{C}\psi_C=\lambda_C\psi_C$, the trace becomes the sum of all eigenvalues. Therefore, in the case of discrete spectrum we'll find

$$\sum_{C=-\infty}^{\infty}\lambda_C=0\,, \qquad (5)$$

whereas for continuous spectrum

$$\sum_{C=-\infty}^{\infty}\int_{-\infty}^{\infty}d\lambda_C=0\,. \qquad (6)$$

There exist operators that obey such requirements automatically. Relevant example is, for instance, angular momentum algebra

$$\left[\hat{L}_i,\hat{L}_j\right]=i\varepsilon_{ijk}\hat{L}_k \qquad (7)$$

or, in particular case,

$$\left[\hat{L}_1,\hat{L}_2\right]=i\hat{L}_3 \qquad (8)$$

for which

$$\sum\lambda_C=\sum_{m=-l}^{l}m=0,\quad m=0,\pm1,\pm2,...,\pm l \qquad (9)$$
$$\left(\text{discrete spectrum}\right)$$

The general transformation rule for vector-like operators

$$\left[\hat{L}_i,\hat{O}_J\right]=i\varepsilon_{ijk}\hat{O}_k \qquad (10)$$

also exhibits such a property. For example, if $\hat{O}_k$ is a coordinate operator $\hat{Q}_k$ or momentum operator $\hat{P}_k$ with eigenvalues $x_k\left(p_k\right)$, it follows that the sum of all coordinates and all momentum eigenvalues will be zero. This occurs for all vectorial operators because of two signs of their eigenvalues.

Physical operators corresponding to observables as a rule, belong to certain Lie algebras, and act as their generators. Therefore, traceless requirement would impose some restrictions on the sum of their eigenvalues.

However, there exist many exceptions as well. Operators with non-symmetric spectra between positive and negative eigenvalues, are not traceless. The best example is the Hamiltonian. It appears on the right-hand side of the commutator with a dilation operator D in the following form [7]:

$$\left[\hat{D},\hat{P}_\mu\right]=-i\hat{P}_\mu\,. \qquad (11)$$

There is no guarantee that these commutation relations remain valid when conformal invariance is broken. However, it is known conterexample, namely, the r^{-2} potential in the Schrödinger equation across any dimension, renders the theory scale invariance, at least formally. Specifically, there exists a dilation operator D [8]

$$\hat{D}=t\hat{H}-\frac{1}{4}\left(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}}+\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}\right), \qquad (12)$$

that it has following commutator with the Hamiltonian

$$i\left[\hat{D},\hat{H}\right]=\hat{H}\,. \qquad (13)$$

Evidently, there is a contradiction in general, $Tr\left(\hat{H}\right)\neq0$, operators with non-symmetric spectra between positive and negative eigenvalues do not have zero traces.

Therefore, we have two-sorts of operators: operators with zero traces and without zero traces. The first-sort of operators can be represented by commutators of two other operators, whereas the second-sort ones are not such. It is unknown for us how these two sorts are alike and how they differ in mathematical structure.

This commutation relation (10) appears in wider 15-parameters Lie group, where together with the Poincare transformations the dilatation and special conformal transformations are also included.

The time component $(\mu = 0)$ in (9) is a Hamiltonian $\hat{H}$, for space components $(\mu = 1, 2, 3)$ it is a linear momentum vector-operator $\hat{\boldsymbol{P}}$. For the last one there is no problem. But for Hamiltonian, zero trace is excluded, $Tr(\hat{H}) \neq 0$ and we meet a contradiction like the above mentioned. It seems, “Something is rotten in the state of Denmark!”

ფიზიკა

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