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Mathematics

Explicit Stochastic Integral Representation of Path-Dependent Brownian Functionals

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We consider some path-dependent Brownian functionals and derive constructive formulas for the stochastic integral representation. The class of functionals under consideration is not stochastically (in Malliavin sense) smooth, and both the well-known Clark-Ocone formula (1984) and its generalization, the Glonti-Purtukhia formula (2017), are inapplicable to them. Here, we use a certain modification of the earlier generalization of the Clark-Ocone formula by Jaoshvili-Purtukhia (2005) and derive a stochastic integral representation with an explicit form of the integrands.

Brownian functional, Malliavin derivative, stochastic integral representation, martingale representation, Clark–Ocone formula

Auxiliary Concepts and Results

The question of representing Brownian functionals as a stochastic Itô integral with an explicit form of the integrand is investigated. In general, the representation of one martingale as a stochastic integral with respect to another martingale, when the martingale is adapted with respect to the natural flow of the σ-algebras of the last martingale, is called a martingale representation. The Martingale Representation Theorem (MRT) provides a powerful framework for understanding and manipulating stochastic processes driven by Brownian motion. It has profound implications across various fields, including finance, physics, and engineering, by providing a deeper understanding of the relationship between martingales and the driving Brownian motion.

On the other hand, it is known that the Itô stochastic integral as a process under certain conditions on the integrand is a martingale with respect to the natural flow of σ-algebras of Brownian motion. The question naturally arises whether the opposite is true: can any martingale with respect to the natural flow of σ-algebras of Brownian motion be represented as a stochastic Itô integral? The well-known theorem of Clark [1] gives a positive answer to this question (which is a particular case of solving the martingale representation problem).

It should be noted that finding an explicit expression for the integrand $\varphi(t,\omega)$ is a very difficult task. In this direction, one general result is known, called the Clark-Ocone formula [2], which is a powerful result in stochastic calculus, giving a representation for square-integrable functionals of Brownian motion in the form of stochastic integrals.

The Clark-Ocone representation has numerous applications in various fields: It plays a crucial role in option pricing, hedging, and portfolio optimization; It is used to solve optimal control problems where the underlying system is driven by Brownian motion; It can be applied to signal processing problems to derive optimal filters for estimating signals corrupted by noise.

According to the Clark-Ocone formula $\varphi(t,\omega)=E\left[D_tF\mid\Im_t\right](\omega)$, where D_tF is the so-called stochastic Malliavin derivative of the functional F under consideration. The Malliavin derivative is a concept from Malliavin calculus, which is a differential calculus on Wiener space. It provides a way to differentiate functionals of Brownian motion.

Let a Brownian Motion $B=(B_t),\ t\in[0,T]$ be given on a probability space $(\Omega,\Im,P)$, and let $\Im_t:=\Im_t^B=\sigma\{B_u:0\le u\le t\}$. The stochastic derivative (derivative in the Malliavin sense) of a smooth random variable $F=f\left(B_{t_1},B_{t_2},...,B_{t_n}\right)\left(f\in C_p^\infty,\ t_i\in\left[0,T\right]\right)$ is defined as a random process D_tF defined by the relation (see, [3])

$$D_tF=\sum_{i=1}^n\frac{\partial f}{\partial x_i}\left(B_{t_1},B_{t_2},...,B_{t_n}\right)I_{[0,t_i]}(t)\,.$$

D is closable as an operator from $L_2(\Omega)$ to $L_2\left(\Omega;L_2\left(\left[0,T\right]\right)\right)$. Denote its domain of definition by $D_{1,2}$. This means that $D_{1,2}$ is equal to the closure of the class of smooth random variables in the norm

$$\|F\|_{1,2}:=\left\{E\left[F^2\right]+E\left[\|DF\|^2_{L_2([0,T])}\right]\right\}^{1/2}.$$

Theorem 1 (see [2]). If F is differentiable in the sense of Malliavin, $F\in D_{1,2}$, then the following stochastic integral representation holds

$$F=E\left[F\right]+\int_0^T E\left[D_tF\mid\Im_t\right]dB_t\quad(P\text{-a.s.}).$$

Shiryaev and Yor [4] and Graversen, Shiryaev and Yor [5] proposed another method for finding the integrand based on the Itô formula and Levy's theorem for the Levy martingale $M_t=E\left[F\mid\Im_t\right]$ associated with the considered functional F (as F they considered the so-called "maximal" type functionals of Brownian Motion).

Later, using the Clark-Ocone formula, Renaud and Remillard [6] established an explicit martingale representation for Brownian functionals, which also depend on the trajectory (in particular, here F is a continuously differentiable function of three smooth quantities: from the Brownian Motion with drift and processes of its maximum and minimum).

Further, it turned out that the requirement for the smoothness of a functional can be weakened by the requirement for the smoothness of only its conditional mathematical expectation. It is known that if a random variable is stochastically differentiable in the sense of Malliavin, then its conditional mathematical expectation is also differentiable.

Glonti and Purtukhia (2017) generalized the Clark-Ocone formula to the case when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable, and proposed a method for finding the integrand.

Theorem 2 (see [7]). Assume that $g_t = E[F \mid \Im_t]$ is a Malliavin differentiable functional $(g_t(\cdot) \in D_{1,2})$. for almost all $t \in [0,T)$.. Then the following stochastic integral representation is valid:

$$g_T = F = E[F] + \int_0^T \nu_u dB_u \quad (P\text{-a.s.}),$$

where

$$\nu_u = \lim_{t \uparrow T} E[D_u g_t \mid \Im_u] \;\text{ in }\; L_2([0,T] \times \Omega).$$

Here we consider some path-dependent Brownian functionals and derive constructive formulas for the stochastic integral representation. The considered class of functionals is not stochastically smooth, and both the well-known Clark-Ocone formula (1984) and its generalization, the Glonti-Purtukhia formula (2017), are not applicable to them. Here we use a certain modification of the earlier Jaoshvili-Purtukhia (2005) generalization of the Clark-Ocone formula, according to which the following result is valid.

Theorem 3 (see [8]). Let f and its generalized derivative ∂f be a square-integrable functions with weight function $\exp\left\{-\frac{x^2}{2}\right\}$ and $\xi \in D_{1,2}$, then the following stochastic integral representation is valid:

$$f(\xi) = E[f(\xi)] + \int_0^T E[\partial f(\xi) D_t \xi \mid \Im_t] dB_t \quad (P\text{-a.s.}).$$

In addition, we needed two technical results about ordinary integrals, which are obviously moments of the standard normal distribution in the case of a complete space $(a = -\infty)$.

Proposition 1. For any real number a and non-negative integer $n \geq 0$ we have

$$\frac{1}{\sqrt{2\pi}} \int_a^{+\infty} x^{2n} \exp\left\{-\frac{x^2}{2}\right\} dx = \varphi(a) \sum_{k=1}^{n} \frac{(2n-1)!!}{(2k-1)!!} a^{2k-1} + (2n-1)!![1 - \Phi(a)],$$

where Φ is the standard normal distribution function and φ is its density function, $(2n-1)!! := 1 \cdot 3 \cdots (2n-1)$, $(-1)!! := 1$.

Proof. To verify the proposition, we will use the method of mathematical induction. For $n = 1$, using the formula of integration by parts, it is obvious that

$$\frac{1}{\sqrt{2\pi}} \int_a^{+\infty} x^2 \exp\left\{-\frac{x^2}{2}\right\} dx = -\frac{1}{\sqrt{2\pi}} \int_a^{+\infty} x d\left(\exp\left\{-\frac{x^2}{2}\right\}\right) =$$

$$= -\frac{x}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} \Big|_a^{+\infty} + \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} \exp\left\{-\frac{x^2}{2}\right\} dx = a\varphi(a) + 1 - \Phi(a).$$

Now suppose that this relation is true for n and check that it is also true for $n+1$. Again, thanks to the integration by parts formula, it is easy to see that

$$\frac{1}{\sqrt{2\pi}}\int_a^{+\infty} x^{2(n+1)}\exp\left\{-\frac{x^2}{2}\right\}dx = -\frac{1}{\sqrt{2\pi}}\int_a^{+\infty} x^{2n+1} d\left(\exp\left\{-\frac{x^2}{2}\right\}\right) =$$

$$= -\frac{x^{2n+1}}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}\Big|_a^{+\infty} + \frac{2n+1}{\sqrt{2\pi}}\int_a^{+\infty} x^{2n}\exp\left\{-\frac{x^2}{2}\right\}dx = \frac{a^{2n+1}}{\sqrt{2\pi}}\exp\left\{-\frac{a^2}{2}\right\} +$$

$$+(2n+1)\left\{\varphi(a)\sum_{k=1}^{n}\frac{(2n-1)!!}{(2k-1)!!}a^{2k-1} + (2n-1)!!\left[1-\Phi(a)\right]\right\} =$$

$$= \varphi(a)\sum_{k=1}^{n+1}\frac{(2n+1)!!}{(2k-1)!!}a^{2k-1} + (2n+1)!!\left[1-\Phi(a)\right].$$

By that, the proof of proposition is complete.

In the same way, the validity of the following proposition is verified.

Proposition 2. For any real number a and natural n we have

$$\frac{1}{\sqrt{2\pi}}\int_a^{+\infty} x^{2n-1}\exp\left\{-\frac{x^2}{2}\right\}dx = \varphi(a)\sum_{k=0}^{n-1}\frac{(2n-2)!!}{(2k)!!}a^{2k},$$

where $(2n)!! := 2\cdot 4\cdots(2n)$, $(0)!! := 1$.

Stochastic integral representation. For any non-negative integer n, consider the following path-dependent Brownian functional:

$$F(n) := \left(\int_0^T B_s ds\right)^{2n+1}.$$

Let us denote $F^+(n) := [F(n)]^+$ and $F^-(n) := \left[F(n)\right]^-$.

Theorem 4. For any non-negative integer n the following stochastic integral representation is valid:

$$F^+(n) = E\left[F^+(n)\right] + (2n+1)\sum_{k=0}^{2n}\int_0^T (T-t)C_{2n}^k \sigma^k \eta^{2n-k} I_k^+(\sigma,\eta)\Big|_{\eta=\int_0^t (T-s)dB_s} dB_t,$$

where

$$I_{2k-1}^+(\sigma,\eta) := \varphi\left(\frac{\eta}{\sigma}\right)\sum_{i=0}^{k-1}\frac{(2k-2)!!}{(2i)!!}\left(\frac{\eta}{\sigma}\right)^{2i},$$

$$I_{2k}^+(\sigma,\eta) := (2k-1)!!\,\Phi\left(\frac{\eta}{\sigma}\right) + \varphi\left(\frac{\eta}{\sigma}\right)\sum_{i=1}^{k}\frac{(2k-1)!!}{(2i-1)!!}\left(-\frac{\eta}{\sigma}\right)^{2i-1}.$$

Proof. Using the formula for integration by parts, we can write

$$\int_0^T B_s ds = sB_s\Big|_0^T - \int_0^T s dB_s = TB_T - \int_0^T s dB_s = \int_0^T (T-s)dB_s.$$

Hence, it's obvious that

$$\int_0^T (T-s)dB_s \sim N\left(0, T^3/3\right)$$

and

$$\int_t^T (T-s)dB_s \sim N\left(0, (T-t)^3/3\right) := N\left(0,\sigma^2\right).$$

According to Theorem 3, using the well-known properties of the Malliavin derivative (see [9]), it is not difficult to see that

$$\varphi^{+}(t,\omega)=E\left[\partial F^{+}(n)D_t\left(\int_0^T B_s ds\right)\mid\Im_t\right]=E\left[I_{\left\{\left(\int_0^T B_s ds\right)^{2n+1}>0\right\}}\left(2n+1\right)\left(\int_0^T B_s ds\right)^{2n}\cdot\int_0^T I_{[0,s]}\left(t\right)ds\mid\Im_t\right]=$$

$$=\left(2n+1\right)\left(T-t\right)E\left[I_{\left\{\int_0^T (T-s)dB_s>0\right\}}\left(\int_0^T\left(T-s\right)dB_s\right)^{2n}\mid\Im_t\right]=\left(2n+1\right)\left(T-t\right)\times$$

$$\times E\left[I_{\left\{\int_t^T (T-s)dB_s+\int_0^t (T-s)dB_s>0\right\}}\left(\int_t^T\left(T-s\right)dB_s+\int_0^t\left(T-s\right)dB_s\right)^{2n}\mid\Im_t\right].$$

Further, thanks to the well-known properties of conditional mathematical expectation, using the Newton binomial formula and the values of the moments of the normal distribution, it is not difficult to establish that

$$\varphi^{+}\left(t,\omega\right)=\left(2n+1\right)\left(T-t\right)E\left[I_{\left\{\int_t^T (T-s)dB_s>-y\right\}}\left(\int_t^T\left(T-s\right)dB_s+y\right)^{2n}\right]\Big|_{y=\int_0^t (T-s)dB_s}=$$

$$=\frac{\left(2n+1\right)\left(T-t\right)}{\sqrt{2\pi}\sigma}\sum_{r=0}^{2n}C_{2n}^{r}y^{2n-r}\int_{-y}^{+\infty}x^{r}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}dx\,\Big|_{y=\int_0^t (T-s)dB_s}=$$

$$=\left(2n+1\right)\left(T-t\right)\sum_{r=0}^{2n}C_{2n}^{r}\sigma^{r}y^{2n-r}I_{2n}^{+}\left(r,\sigma,y\right)\Big|_{y=\int_0^t (T-s)dB_s},$$

where

$$I_{2n}^{+}\left(r,\sigma,y\right):=\frac{1}{\sqrt{2\pi}}\int_{-y/\sigma}^{+\infty}x^{r}\exp\left\{-\frac{x^2}{2}\right\}dx.$$

On the other hand, from Proposition 1 and 2, for any natural r we have

$$I_{2n}^{+}\left(2r-1,\sigma,y\right):=\frac{1}{\sqrt{2\pi}}\int_{-y/\sigma}^{+\infty}x^{2r-1}\exp\left\{-\frac{x^2}{2}\right\}dx=$$

$$=\varphi\left(-\frac{y}{\sigma}\right)\sum_{k=0}^{r-1}\frac{\left(2r-2\right)!!}{\left(2k\right)!!}\left(-\frac{y}{\sigma}\right)^{2k}=\varphi\left(\frac{y}{\sigma}\right)\sum_{k=0}^{r-1}\frac{\left(2r-2\right)!!}{\left(2k\right)!!}\left(\frac{y}{\sigma}\right)^{2k}$$

and

$$I_{2n}^{+}\left(2r,\sigma,y\right):=\frac{1}{\sqrt{2\pi}}\int_{-y/\sigma}^{+\infty}x^{2r}\exp\left\{-\frac{x^2}{2}\right\}dx=$$

$$=\varphi\left(-\frac{y}{\sigma}\right)\sum_{k=1}^{r}\frac{\left(2r-1\right)!!}{\left(2k-1\right)!!}\left(-\frac{y}{\sigma}\right)^{2k-1}+\left(2r-1\right)!!\left[1-\Phi\left(-y/\sigma\right)\right]=$$

$$=\varphi\left(\frac{y}{\sigma}\right)\sum_{k=1}^{r}\frac{\left(2r-1\right)!!}{\left(2k-1\right)!!}\left(-\frac{y}{\sigma}\right)^{2k-1}+\left(2r-1\right)!!\Phi\left(y/\sigma\right).$$

Using similar arguments, the following result is proven.

Theorem 5. For any non-negative integer n the following stochastic integral representation is valid:

$$F^{-}(n) = E\left[F^{-}(n)\right] - (2n+1)\sum_{k=0}^{2n}\int_0^T (T-t)C_{2n}^k \sigma^k y^{2n-k} I_k^{-}(\sigma, y)|_{y=\int_0^t (T-s)dB_s} \; dB_t,$$

where

$$I_{2k-1}^{-}(\sigma, y) := -\varphi\left(\frac{\eta}{\sigma}\right)\sum_{i=0}^{k-1}\frac{(2k-2)!!}{(2i)!!}\left(\frac{\eta}{\sigma}\right)^{2i},$$

$$I_{2k}^{-}(\sigma, y) := (2k-1)!!\left[1-\Phi\left(\frac{\eta}{\sigma}\right)\right] - \varphi\left(\frac{\eta}{\sigma}\right)\sum_{i=1}^{k}\frac{(2k-1)!!}{(2i-1)!!}\left(-\frac{\eta}{\sigma}\right)^{2i-1}.$$

Combining Theorems 4 and 5, we obtain the following result.

Theorem 6. For any non-negative integer n the following stochastic integral representation is valid:

$$\left|F(n)\right| = E\left|F(n)\right| + (2n+1)\sum_{k=0}^{2n}\int_0^T (T-t)C_{2n}^k \sigma^k \eta^{2n-k} I_k(\sigma,\eta)|_{\eta=\int_0^t (T-s)dB_s} \; dB_t,$$

where

$$I_{2k-1}(\sigma, y) := 2\varphi\left(\frac{\eta}{\sigma}\right)\sum_{i=0}^{k-1}\frac{(2k-2)!!}{(2i)!!}\left(\frac{\eta}{\sigma}\right)^{2i},$$

$$I_{2k}(\sigma, y) := (2k-1)!!\left[2\Phi\left(\frac{\eta}{\sigma}\right)-1\right] + 2\varphi\left(\frac{\eta}{\sigma}\right)\sum_{i=1}^{k}\frac{(2k-1)!!}{(2i-1)!!}\left(-\frac{\eta}{\sigma}\right)^{2i-1}.$$

Corollary 2.1. The following stochastic integral representation is true

$$\left|\int_0^T B_s ds\right| = \sigma\sqrt{\frac{2}{\pi}} + \int_0^T (T-t)\left[2\Phi\left(\frac{1}{\sigma}\int_0^t (T-s)dB_s\right)-1\right]dB_t.$$

მათემატიკა

ტრაექტორიაზე დამოკიდებული ბროუნის ფუნქციონალების სტოქასტური ინტეგრალური წარმოდგენა ცხადი სახით

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