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Mathematics

Weighted Multilinear Rellich and Hardy Inequalities on Homogeneous Groups

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Abstract. Two-weight multilinear Rellich and Hardy differential inequalities are established, generally speaking, on nilpotent Lie groups G. Multilinear Rellich inequalities in the one-dimensional case were previously derived by the authors of this paper in 2021, but in the higher dimensional case the problem remained open. The results presented in this work are novel even for both the Abelian (Euclidean) case, and the Heisenberg groups $G = \boldsymbol{H}^n$. *© 2025 Bull. Georg. Natl. Acad. Sci.*

Keywords: multilinear Rellich inequality, multilinear Hardy inequality, weighted inequality, homogeneous groups, Heisenberg groups

Introduction

Rellich's classical inequality states that if $u \in C_0^\infty(\boldsymbol{R}^d\backslash\{0\})$ and $d \neq 2$, then

$$\frac{d^2(d-4)^2}{16}\int_{R^d}\frac{|u(x)|^2}{|x|^4}dx \leq \int_{\boldsymbol{R}^d}|\Delta u(x)|^2 dx\,,$$

and the constant $d^2(d-4)^2/16$ is sharp. This inequality was announced by Rellich in 1954 [1]. When $d = 2$ the inequality still holds but only for a restricted class of functions [2].

For further progress regarding Rellich-type inequalities in the classical Lebesgue spaces we refer e.g., to [2-4], and references cited therein.

In [5], the authors established the one-dimensional multilinear Rellich inequality:

$$\left(\int_a^b w(\delta(x))\left|\prod_{k=1}^m u_k(x)\right|^q dx\right)^{\frac{1}{q}} \leq C\prod_{k=1}^m \|u_k''\|_{L^{p_k(a,b)}},\ u_k \in C_0^2(a,b),\ k = 1,\dots,m,$$

with a positive function w on $(0, b-a)$, m is a positive integer, $\delta(x) = min\{x-a, b-x\}$ is the distance function on (a,b).

Our aim is to establish weighted higher-dimensional Rellich and Hardy differential inequalities:

$$\left(\int_G v(x)\left|\prod_{k=1}^m u_k(x)\right|^q dx\right)^{\frac{1}{q}} \leq C\prod_{k=1}^m\left(\int_G w_k(x)|\Delta_G u_k|^{p_k}dx\right)^{\frac{1}{p_k}}$$

and

$$\left(\int_G v(x)\left|\prod_{k=1}^{m} u_k(x)\right|^q dx\right)^{\frac{1}{q}} \le C\prod_{k=1}^{m}\left(\left(\int_G w_k(x)|\nabla_G u_k|^{p_k}dx\right)^{\frac{1}{p_k}}\right),$$

where v is a general weight, and w_k are power-type weights.

The problem is studied on nilpotent Lie groups but the results are new even for the Abelian (Euclidean) case, and the Heisenberg groups $G = \boldsymbol{H}^n$.

Now we recall definitions and some of the main properties of (stratified) homogeneous groups [6-10].

Definition 1.1. A Lie group (on $\boldsymbol{R}^d$) G is said to be homogeneous if there is a dilation $D_\lambda(x)$ such that

$$D_\lambda(x) := (\lambda^{v_1}x_1, \dots, \lambda^{v_d}x_d),\ v_1, \dots, v_d > 0,\ D_\lambda: \boldsymbol{R}^d \to \boldsymbol{R}^d,$$

which is an automorphism of the group G for each $\lambda > 0$. Throughout this paper we use the notation λx for the dilation $D_\lambda(x)$. The number $Q := v_1 + \dots + v_d$ is called the homogeneous dimension of G (in the Euclidean case we have $Q = d$). A homogeneous quasi-norm on G is a continuous non-negative function $r: G \to [0, \infty)$ such that

i) $r(x) = r(x^{-1})$ for all $x \in G$,
ii) $r(\lambda x) = \lambda r(x)$ for all $x \in G$ and $\lambda > 0$,
iii) $r(x) = 0$ if and only $x = 0$.

The quasi-ball centred at $x \in G$ with radius $R > 0$ is defined by

$$B(x, R) := \{y \in G : r(x^{-1}y) < R\}.$$

A homogeneous group is necessarily nilpotent, and the Haar measure on G coincides with the Lebesgue measure; we will denote it by dx. If $|E|$ denotes the measure of a measurable set $E \subset G$, then

$$|D_\lambda(E)| = \lambda^Q |E| \quad \text{and} \quad \int_G f(\lambda x)dx = \lambda^{-Q}\int_G f(x)dx.$$

Hence, we have that the Haar measure of the quasi-ball has the following property: there is a constant $A \ge 1$ such that

$$A^{-1}R^Q \le |B(x, R)| \le AR^Q.$$

A homogeneous group with the quasi-norm $r(\cdot)$ and Haar measure dx is an example of a quasi-metric measure space with a doubling measure, which is also called a space of homogeneous type (*SHT* briefly).

Let us now recall the definition of a homogeneous stratified group (or homogeneous Carnot group). These form an important class of homogeneous groups. We refer, e.g., to [6,7,9].

Definition 1.2. A Lie group $G = (\boldsymbol{R}^d, \circ)$, is called a homogeneous stratified group if the following conditions hold:

i) the decomposition $\boldsymbol{R}^d = \boldsymbol{R}^{d_1} \times \dots \times \boldsymbol{R}^{d_s}$ is valid for some natural numbers $d_1, \dots, d_s$ with $d_1 + \dots + d_s = d$; the dilation $\delta_\lambda: \boldsymbol{R}^d \to \boldsymbol{R}^d$ given by

$$\delta_\lambda(x) \equiv \delta_\lambda(x^{(1)}, \dots, x^{(s)}) := (\lambda x^{(1)}, \dots, \lambda^s x^{(s)}),\ x^{(k)} \in \boldsymbol{R}^{d_k},\ k = 1, \dots, s,$$

is an automorphism of the group G for every $\lambda > 0$.

ii) If d_1 is as in (a) and $X_1, \dots, X_{d_1}$ are the left-invariant vector fields on G such that $X_k(0) = \left.\frac{\partial}{\partial x_k}\right|_0$ for $k = 1, \dots, d_1$, then

$$rank\big(Lie\{X_1, \dots, X_{d_1}\}\big) = d,$$

for every $x \in \boldsymbol{R}^d$. In another words, the iterated commutators of $X_1, \dots, X_{d_1}$ span the Lie algebra of G.

In the sequel, by the symbol

$$\nabla_G := \left(X_1, \dots, X_{d_1}\right)$$

we denote the horizontal gradient on G. Hence, the sub-Laplacian on (homogeneous) stratified groups is determined by the formula

$$\Delta_G := \nabla_G \cdot \nabla_G$$

In general, a Lie group is called stratified if it is a connected and simply-connected Lie group whose Lie algebra is stratified. Any (abstract) stratified group is isomorphic to a homogeneous one.

We will assume that G is a stratified homogeneous group.

A locally integrable a.e. positive function on G is called a weight. For a weight function w, we denote by $L^p_w(G)$, for $1 < p < \infty$, the weighted Lebesgue space defined with respect to the standard norm

$$\|f\|_{L^p_w(G)} := \left(\int_G |f(x)|^p w(x) dx\right)^{\frac{1}{p}}.$$

If w is a constant, then $L^p_w(G)$ will be denoted by $L^P(G)$.

Two-weight one-dimensional Rellich inequalities were derived in [11] in the linear case.

In the sequel we will use the notation

$$\frac{1}{p} := \sum_{k=1}^{m} \frac{1}{p_k}.$$

Main Results

Now we formulate the main results of this note:

Theorem 1. (Weighted Rellich Inequality). Let G be a stratified homogeneous group with homogeneous norm $r(x)$ and homogeneous dimension $Q > 2$. Let $\frac{Q}{2} < min\{p_1, \dots, p_m\} \le max\{p_1, \dots, p_m\} \le q < \infty$. Suppose that $2p_i - Q < \beta_i < Q(p_i - 1)$, $i = 1, \dots, m$. Let v be a weight function on G. Then the condition

$$\sup_{k\in\mathbb{Z}} \left(\int_{B(0,2^{k+1})\setminus B(0,2^k)} v(x) dx\right)^{\frac{1}{q}} 2^{k\left(2m - \frac{Q}{p} - \sum_{j=1}^{m} \frac{\beta_j}{p_j}\right)} < \infty$$

implied the following inequality

$$\|\prod_{k=1}^{m} u_k\|_{L^q_v(G)} \le C \prod_{k=1}^{m} \|\Delta u_k\|_{L^{p_k}_{r(x)}(G)}, \; u_k \in C_0^\infty(\boldsymbol{R}^d, \boldsymbol{R}), \qquad k = 1, \dots, m,$$

with the positive constant C independent of u_k, $k = 1, \dots, m$.

Theorem 2. (Weighted Hardy Inequality). Let G be a stratified homogeneous group. Let $Q < min\{p_1, \dots, p_m\} \le max\{p_1, \dots, p_m\} \le q < \infty$. Suppose that $p_i - Q < \beta_i < Q(p_i - 1)$, $i = 1, \dots, m$. Let v be a weight function on G. Then the condition

$$\sup_{k\in\mathbb{Z}} \left(\int_{B(0,2^{k+1})\setminus B(0,2^k)} v(x) dx\right)^{\frac{1}{q}} 2^{k\left(m - \frac{Q}{p} - \sum_{j=1}^{m} \frac{\beta_j}{p_j}\right)} < \infty$$

implied the following inequality

$$\|\prod_{k=1}^{m} u_k\|_{L^q_v(G)} \le C \prod_{k=1}^{m} \|\nabla_G u_k\|_{L^{p_k}_{r(x)^{\beta_k}}(G)}, \; u_k \in C_0^\infty(\boldsymbol{R}^d, \boldsymbol{R}), \; k = 1, \dots, m,$$

with the positive constant C independent of u_k, $k=1,\dots,m$.

As a special case we have the following statements with power weights:

Theorem 3. (Weighted Rellich Inequality with power weights). Let G be a stratified homogeneous group with homogeneous dimension $Q>2$. Let $\frac{Q}{2}<min\{p_1,\dots,p_m\}\le max\{p_1,\dots,p_m\}\le q<\infty$. Suppose that $2p_i-Q<\beta_i<Q(p_i-1)$, $i=1,\dots,m$, and that $\gamma=q\left(\sum_{j=1}^{m}\frac{\beta_j}{p_j}\right)-2mq+\frac{qQ}{p}-Q$. Then there exists a positive constant C such that for all $u_k\in C_0^\infty(\boldsymbol{R}^d,\boldsymbol{R})$, $k=1,\dots,m$. the inequality

$$\left\|\prod_{k=1}^{m}u_k\right\|_{L^q_{r(x)^\gamma}(G)}\le C\prod_{k=1}^{m}\|\Delta_G\, u_k\|_{L^{p_k}_{r(x)^{\beta_k}}(G)}$$

holds.

Theorem 4. (Weighted Hardy Inequality with power weights). Let G be a stratified homogeneous group. Let $Q<min\{p_1,\dots,p_m\}\le max\{p_1,\dots,p_m\}\le q<\infty$. Suppose that $p_i-Q<\beta_i<Q(p_i-1)$, $i=1,\dots,m$, and that $\gamma=q\left(\sum_{j=1}^{m}\frac{\beta_j}{p_j}\right)-mq+\frac{qQ}{p}-Q$. Then there exists a positive constant C such that for all $u_k\in C_0^\infty(\boldsymbol{R}^d,\boldsymbol{R})$, $k=1,\dots,m$. the inequality

$$\left\|\prod_{k=1}^{m}u_k\right\|_{L^q_{r(x)^\gamma}(G)}\le C\prod_{k=1}^{m}\|\nabla_G u_k\|_{L^{p_k}_{r(x)^{\beta_k}}(G)}$$

holds.

მათემატიკა

წონიანი მრავალწრფივი რელიხისა და ჰარდის უტოლობები ერთგვაროვან ჯგუფებზე

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