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Physics

A Field-Theoretical Model with Saddle Point Configuration

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Abstract. This study explores an extended σ-model in (3+1)-dimensions, enhanced by the addition of higher-order terms in kinetic energy – specifically anti-symmetric combinations of derivatives of the fields of fourth and sixth powers. Such terms enable localized field configurations with finite energy to exist whereas theories with the quadratic combination of field derivatives do not in more than (1+1)-dimensions based on the virial theorem. The potential is chosen to break the symmetry of the theory concerning the orthogonal group O(5). We focus on sphaleron and instanton configurations of the classical field equations of motion in Minkowski space. To obtain explicit solutions, a specific symmetry-breaking potential is selected. Since the equations are too complex to solve directly, the Lagrangian is simplified by retaining only the sextic term in the kinetic energy. With this simplification, explicit expressions for the instanton and sphaleron are derived, and their stability equations are studied. The stability equations are further simplified to identify solutions with negative eigenvalues, demonstrating the instability of the sphaleron configuration. An explicit expression for the classical energy of the sphaleron is obtained, along with confirmation of a single negative eigenvalue in the fluctuation equations, proving the existence of the sphaleron as an unstable field configuration. *© 2025 Bull. Georg. Natl. Acad. Sci.*

Keywords: σ-model, sphaleron, instanton, stability

Introduction

The saddle-point configurations in classical field theories have attracted attention as a mechanism for baryon-number violation conservation [1-3]. It's difficult to explicitly find such a configuration in real theories, like electroweak theory. Much simpler models in lower dimensions, like σ-model [4] or scalar φ^4-theory [5-8] on a circle are often studied. The properties of a class of such models in (2+1)-dimensions were investigated in [9-11]. It is known that the sphaleron energy is interpreted as the potential barrier height for tunneling between topologically distinct vacua.

In this paper, we investigate a (3+1)-dimensional modified σ-model, in particular, O(5)-invariant theory extended by the addition of kinetic terms of higher order, indeed quartic and sextic terms that allow finite energy classical field configurations in real space.

We consider five real scalar fields $\varphi_a(x)$, $a=1\ldots5$ in (3+1)-dimensions subject to constraint $\sum_a \varphi_a^2(x)=1$. The Lagrangian which allows finite energy classical field configuration is choosen as follows:

$$\mathcal{L}=\lambda_0\Phi_{\mu\nu\sigma,abc}\Phi^{\mu\nu\sigma}_{abc}-\lambda_1\Phi_{\mu\nu,ab}\Phi^{\mu\nu}_{ab}+\lambda_2\partial_\mu\varphi_a(\boldsymbol{x},t)\partial^\mu\varphi_a(\boldsymbol{x},t)+U(\lambda,\varphi(\boldsymbol{x},t)), \tag{1}$$

in which the following notations are used (the metric is $\eta_\mu \nu=(+,-,-,-)$)

$$\Phi_{\mu\nu,ab}=\partial_\mu\varphi_a(\boldsymbol{x},t)\partial_\nu\varphi_b(\boldsymbol{x},t)-\partial_\nu\varphi_b(\boldsymbol{x},t)\partial_\mu\varphi_a(\boldsymbol{x},t), \tag{2}$$

$$\Phi_{\mu\nu\sigma,abc}=\partial_\mu\varphi_a(\boldsymbol{x},t)\Phi_{\nu\sigma,bc}$$

+ all terms obtained by cyclic permutations of all indices. (3)

These terms are fully antisymmetric in their indices. As we are interested in both instanton and sphaleron configurations, we start from the Euclidian d-dimensional space. Based on scaling arguments, we can see (apart from the potential term) that

$$\lambda_0(d-6)\left\|\Phi_{\mu\nu\sigma,abc}\Phi^{\mu\nu\sigma}_{abc}\right\|+\lambda_1\left\|\Phi_{\mu\nu,ab}\Phi^{\mu\nu}_{ab}\right\|+\lambda_2\left\|\partial_\mu\varphi_a(\boldsymbol{x},t)\partial^\mu\varphi_a(\boldsymbol{x},t)\right\|=0. \tag{4}$$

It is evident that the sextic term must be added to the Lagrangian to obtain finite-energy field configurations in 4-dimensional space. The existence of topological stability of an instanton requires fulfillment of another condition. Following [12] and using the inequality (which follows from (4))

$$\left(\lambda_0^{\frac{1}{2}}\Phi_{\mu\nu\sigma,abc}-\lambda_2^{\frac{1}{2}}\varepsilon_{\mu\nu\sigma\varrho}\varepsilon_{abcde}\partial^\varrho\varphi_d\varphi_e\right)\left(\lambda_0^{\frac{1}{2}}\Phi_{\mu\nu\sigma,abc}-\lambda_2^{\frac{1}{2}}\varepsilon_{\mu\nu\sigma\varrho}\varepsilon_{abc\kappa l}\partial^\gamma\varphi_k\varphi_e\right), \tag{5}$$
$$+\lambda_1\Phi_{\mu\nu,ab}\Phi^{\mu\nu}_{ab}\geq 0$$

we obtain the lower bound to the instanton action (taking into account the constraints $\varphi_a\varphi_a=1$, $\varphi_a\partial_\mu\varphi_a=0$):

$$S=\int d^4x\mathcal{L}\geq 2\sqrt{\lambda_0\lambda_2}\int d^4x\varepsilon_{\mu\nu\sigma\varrho}\varepsilon_{abcd}\partial_\mu\varphi_a\partial_\nu\varphi_b\partial_\sigma\varphi_c\partial_\varrho\varphi_d\varphi_e=\frac{8}{3}4!\sqrt{\lambda_0\lambda_2}q, \tag{6}$$

where q is a topological charge. Note that the Euclidean action can be considered as an action in the static limit of a (4+1)-dimensional space. So, we may choose the "hedgehog" ansatz for the instanton configuration:

$$\varphi_\alpha=\left(\cos(fr),\ n_i\sin(f(r))\right) \tag{7}$$

with $r^2=\sum_{i=1}^4 x_i^2$, and $n_i=(\sin\beta\sin\vartheta\cos n\varphi,\ \sin\beta\sin\vartheta\sin n\varphi,\ \sin\beta\cos\vartheta,\cos\beta)$ is a 4-dimensional unit vector with an integer n.

The Euclidian action and the topological charge with this ansatz become:

$$S=2\pi^2\int_0^\infty drr^3\left\{\lambda_2\left(f'^2(r)+3\frac{\sin^2 f(r)}{r^2}\right)+12\lambda_1\frac{\sin^2 f(r)}{r^2}\left(f'^2(r)+\frac{\sin^2 f(r)}{r^2}\right)\right\} \tag{8}$$
$$+628\lambda_0\pi^2\int_0^\infty \mathrm{d}rr^3\frac{\sin^4 f(r)}{r^4}\left(3f'^2(r)+\frac{\sin^2 f(r)}{r^2}\right)$$

$$q=\frac{3}{4}n\int_0^\infty drf'(r)\sin^3 f(r)=\frac{3}{4}n\left.\left(\frac{\cos^3 f(r)}{3}-\cos f(r)\right)\right|_0^\infty \tag{9}$$

The boundary conditions consistent with the finite energy action and nonzero topological charge are given by

$$f(r) \to (2N+1)\pi \quad \text{as} \quad r \to \infty, \qquad f(r) \to 2N\pi \quad \text{as} \quad r \to 0.$$

We now consider the Lagrangian (1) in the Minkowski space. We are interested in static finite energy configuration. A specific symmetry-breaking potential is choosen

$$U(\varphi_a(\boldsymbol{x},t),\lambda) = \lambda^2\sigma^4(1+\varphi_5(\boldsymbol{x},t)^{3k}, \tag{10}$$

where σ is the scaling parameter with the dimension of mass, λ is a dimensionless constant, and k – a nonzero positive integer. Let's introduce a one-parameter family of fields labeled by $\eta \in [0,\pi]$

$$\varphi_a = (-\sin\eta\sin f(r)\boldsymbol{n}, -\sin\eta\cos\eta(1+\cos f(r), \sin^2\eta\cos f(r) - \cos^2\eta), \tag{11}$$

where $\boldsymbol{n} = (\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi, \cos\vartheta)$ is a 3-dimensional unit vector. We are looking for a non-contractible loop (like in [2]) in configuration space that starts and ends at the vacuum $(0,0,0,0,-1)$. One can verify that the fields (11) belong to the sector with the topological number $q = 1$. Thus, the fields at fixed η are maps S^3 to S^4 and the loops are non-contractible and can be considered as a barrier for instanton tunneling, with the sphaleron at the top The energy functional for fixed η with the following choice of parameters $\lambda_0 = \frac{1}{324}\sigma^2$, $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{\sigma^2}{2}$ is defined by ($r^2 = \sum_{i=1}^{3} x_i^2$) is

$$\begin{aligned} E = {} & 4\pi\sin^2\eta\int_0^\infty drr^2\left\{\frac{1}{\sigma^2}\frac{\sin^4 f(r)}{r^4}f'^2(r) + \sin^2\eta\frac{\sin^2 f(r)}{r^2}\left(2f'^2(r) + \frac{\sin^2 f(r)}{r^2}\right)\right\} \\ & + 4\pi\sin^2\eta\int_0^\infty drr^2\left\{\frac{\sigma^2}{2}\left(2f'^2(r) + \frac{\sin^2 f(r)}{r^2}\right) + \lambda^2\sigma^4\sin^{6k-2}\eta(1+\cos f(r))^{3k}\right\}, \end{aligned} \tag{12}$$

which is positive finite functional. Extremizing it we can find the sphaleron configuration. Given the complexity of corresponding equations, we'll simplify our model by setting $\lambda_1 = \lambda_2 = 0$ (note that the stability equation is insensitive to such approximation. Now, the energy functional is defined by equality:

$$E = 4\pi\sin^2\eta\int_0^\infty drr^2\left\{\frac{\sin^4 f(r)}{\sigma^2 r^4}f'^2(r) + \lambda^2\sigma^4\sin^{6k-2}\eta(1+\cos f(r))^{3k}\right\}. \tag{13}$$

By extremizing (13) relative η we'll see that the maximum energy exists at $\eta = \frac{\pi}{2}$. When minimize it by $f(r)$, we obtain the following equation:

$$\frac{\sin^2 f(r)}{r^2}\frac{d}{dr}\frac{\sin^2 f(r)}{r^2}\frac{df(r)}{dr} + \frac{3}{2}k\lambda^2\sigma^6\sin f(r)(1+\cos f(r))^{3k-1} = 0. \tag{14}$$

We have found the solutions of the equations for various values of k, although the explicit expression of the function $f(r)$ is still difficult except $k = 2$. It is found that

$$f(r) = 2\tan^{-1}\sqrt[3]{\lambda}\,\sigma r.$$

Nevertheless, one can integrate the energy by the classical solutions of the equation of motion. So, the energy of the sphaleron (13) is then

$$E = 2^{3k/2+4}\frac{\Gamma(\frac{3\kappa+3}{3})}{\Gamma(\frac{3\kappa}{2}+3)}. \tag{15}$$

One sees that it is positively determined.

To answer the question regarding the stability of the sphaleron, small fluctuations with time around this configuration have to be studied. It turns out that the stability equations are simpler in a different parametrization of the fields [4,13]:

$$\varphi_a = \frac{1}{\sqrt{1+u^2(\boldsymbol{x},t)}}(-\sin\mu(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x},t),\ u(\boldsymbol{x},t), \cos\mu(\boldsymbol{x},t)) \tag{16}$$

with $\boldsymbol{n}(\boldsymbol{x},t) = (\sin v(\boldsymbol{x},t)\cos w(\boldsymbol{x},t), \sin v(\boldsymbol{x},t)\sin w(\boldsymbol{x},t),\ \cos v(\boldsymbol{x},t))$. Next, we choose

$$u(\boldsymbol{x},t) = \psi_1(r,\vartheta,\varphi)\exp(i\omega t), \qquad \mu(\boldsymbol{x},t) = f(r) + \psi_2(r,\vartheta,\varphi)\exp(i\omega t),$$
$$v(\boldsymbol{x},t) = \Theta + \psi_3(r,\vartheta,\varphi)\exp(i\omega t), \qquad u(\boldsymbol{x},t) = \Phi + \psi_4(r,\vartheta,\varphi)\exp(i\omega t). \tag{17}$$

Here the $f(r),\Theta,\Phi$ define the sphaleron, ω^2 determines the eigenvalues of the stability equation and must be negative for instability. The expressions of the stability equations are too cumbersome. This is the system of coupled equations for functions $\psi_i(r,\vartheta,\varphi)$ $(i = 1...4)$, except the equation for ψ_1 which is decoupled from the other three. This equation describes fluctuations in the direction perpendicular to the sphaleron whereas the other three describe fluctuations along the sphaleron ($\eta = \pi/2$). The equation for $\psi_1(r,\vartheta,\varphi)$ is as follows

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\sin^4 f(r)}{r^2}\frac{\partial\psi_1(r,\vartheta,\varphi)}{\partial r}\right) + \frac{\sin^2 f(r)}{r^4}f'^2(r)\,\Delta_\Omega\psi_1(r,\vartheta,\varphi)$$
$$+\lambda^2\sigma^6(1+\cos f(r))^{3k-1}\left(3+\left(3+\frac{3k}{2}\right)\cos f(r)\right)\psi_1(r,\vartheta,\varphi) \tag{18}$$
$$+\omega^2\left\{\frac{\sin^2 f(r)}{r^4}f'^2(r) + \frac{\sin^4 f(r)}{r^4}\right\}\psi_1(r,\vartheta,\varphi) = 0,$$

where Δ_Ω is the angular part of the Laplace operator. We will now show with a reasonable approximation that this equation has exactly one negative value, which indicates a decrease in energy along the vertical direction. Under the spherical symmetry, we can make the following substitution,

$$\psi_1(r,\vartheta,\varphi) = g_1(r)Y_{lm}(\vartheta,\varphi), \tag{19}$$

with $Y_{lm}(\vartheta,\varphi)$ – the well-known spherical harmonics:

$$\Delta_\Omega Y_{lm}(\vartheta,\varphi) = -l(l+1)Y_{lm}(\vartheta,\varphi).$$

By further changing the variables

$$1+\cos f(r) = 2x,\ \ g_1(r) = x^p(1-x)^q h_1(x), \tag{20}$$

the equation becomes the hypergeometric type

$$x(1-x)h_1''(x) + [c-(a+b+1)x]h_1'(x) - abh_1(x) = 0, \tag{21}$$

where $a = p+q-1$, $b = p+q+\frac{3k+6}{2}$, $c = 2p+\frac{3}{0}(k+1)$. The quantities p and q are solutions of the following algebraic equations:

$$\left(p+\frac{3k+1}{4}\right)^2 = \frac{l(l+1)}{4} + \frac{3k}{4} + \frac{(3k+1)^2}{16}, \tag{22}$$

$$\left(q+\frac{1}{4}\right)^2 = \frac{(2l+1)^2}{16}. \tag{23}$$

It can be shown that equation (18) does not have regular solutions for bound states (which we consider) except a constant. This happens for $p = q = \frac{1}{2}$, for which $l = 1$. Hence, the zero eigenvalue exists in the p-wave. There must exist a single physical state with an energy lower than zero. This proves the existence of a negative eigenvalue. Next, we try to calculate this negative eigenmode for $k = 1$ with a reasonable approximation. Consider the expression multiplied by ω^2 in the equation (18). Using the substitutions (19) and (20) one can see that this expression varies slowly with x except near the endpoints $x = 0$, and $x = 1$ where it goes to infinity. Replacing it with the constant $1/(\lambda^{3/2}\sigma^2)$, the equation then becomes hypergeometric like (21) with the parameters

$$a = p + q + 1, \ b = p + q + \frac{9}{2}, c = 2p + 3,$$

where p and q are now solutions of the equations:

$$(p+1)^2 = \frac{l(l+1)+7}{4} - \frac{\omega^2}{\lambda^{\frac{2}{3}}\sigma^2}, \tag{24}$$

$$\left(q + \frac{1}{4}\right)^2 = \frac{l(l+1)}{4} + \frac{1}{16} - \frac{\omega^2}{\lambda^{2/3}\sigma^2}. \tag{25}$$

Based on the same arguments we will find that the only acceptable solution is a constant, for which $p + q - 1 = 0$, hence

$$\omega^2 = \frac{\lambda^{2/3}\sigma^2}{4}[l(l+1) - 2]. \tag{26}$$

We see that the zero eigenvalue still exists for $l = 1$. But there is a negative eigenmode for $l = 0$ which is

$$\omega^2 = -\frac{\lambda^{\frac{2}{3}}\sigma^2}{2}. \tag{27}$$

This is in agreement with the previous case. We did not deal with other stability equations as we have shown the existence of a single negative eigenmode for the stability equations.

ფიზიკა

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ა. შურღაია

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(წარმოდგენილია აკადემიის წევრის ა. კვინიხიძის მიერ)

ნაშრომში განხილულია გაფართოებული σ-მოდელი (3+1)-განზომილებიან სივრცეში, რომელშიც კინეტიკურ ენერგიას ემატება მაღალი რიგის, კერძოდ, ველების მეოთხე და მეექვსე რიგის წარმოებულების ანტისიმეტრიული კომბინაციების მეოთხე და მეექვსე ხარისხის წევრები. ასეთი წევრების არსებობა თეორიაში შესაძლებელს ხდის სასრული ენერგიების მქონე ველების სივრცეში განფენილი კონფიგურაციების არსებობას, მაშინ როდესაც მათ გარეშე ასეთი კონფიგურაციების არსებობას ვირიალის თეორემა უშვებს არა უმეტეს ორგანზომილებიან სივრცეში. პოტენციალი შერჩეულია ისე, რომ დაარღვიოს თეორიის ინვარიანტობა ორთოგონალური O(5) ჯგუფის მიმართ. ყურადღება გამახვილებულია ველის კლასიკური განტოლებების სფალერონისა და ინსტანტონის ტიპის კონფიგურაციებზე მინკოვსკის სივრცეში. სპეციფიკური სიმეტრიის დამრღვევი პოტენციალის შერჩევით მიიღება განტოლებები, რომელთა ამოხსნა შეუძლებელია. ამიტომ ლაგრანჟიანი გამარტივებულია, კერძოდ, მხოლოდ მეექვსე ხარისხის კინეტიკური წევრებია შენარჩუნებული. ამ გამარტივების შემდეგ მიიღება განტოლება, რომლისთვის ნაპოვნია ინსტანტონისა და სფალერონის ტიპის ზუსტი ამოხსნები, მიღებული და შესწავლილია მათი მდგრადობის განტოლებები დროითი შეშფოთების მიმართ. ეს განტოლებები შემდგომში გამარტივებულია გონივრულ მიახლოებაში, რათა გამოვლინდეს უარყოფითი საკუთარი მნიშვნელობების მქონე ამოხსნები, რაც სფალერონის კონფიგურაციის არამდგრადობას ადასტურებს. მიღებულია სფალერონის კლასიკური ენერგიის ცხადი გამოსახულება, აგრეთვე ნაჩვენებია ერთი უარყოფითი საკუთარი მნიშვნელობის არსებობა, რომლისთვისაც მიღებულია ცხადი გამოსახულება, რითაც დასტურდება სფალერონის, როგორც ველის არამდგრადი კონფიგურაციის არსებობა.

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