საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 19(193), №3, 2025
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 19(193), no. 3, 2025

Mathematics

On High-Order Time Differential Dual-Phase-Lag Models of Thermoelastic Solids

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Abstract. In the present paper, a high-order time-differential approximation of the dual-phase-lag dynamic three-dimensional model of thermoelastic solids is considered. The initial-boundary value problem with mixed boundary conditions corresponding to the nonclassical model with two phase-lag parameters is studied, where on certain parts of the boundary displacement and temperature vanish, and on the corresponding remaining parts of the boundary the densities of surface force and heat flux along the outward normal vector are given. A variational formulation of the initial-boundary value problem is obtained, and the existence and uniqueness results and the continuous dependence of the solution on the given data in suitable spaces of vector-valued distributions are proved. © *2025 Bull. Georg. Natl. Acad. Sci.*

Keywords: phase-lag thermoelasticity, initial-boundary value problem, existence, uniqueness, continuous dependence

Introduction

The dual-phase-lag model was proposed by D. Tzou (Tzou, 1995a) to incorporate the effects of microstructural interactions in the fast-transient process of heat transport. Tzou introduced two phase lags for the heat flux vector and the temperature gradient to describe the microstructural effects by the delayed response in time in the macroscopic formulation, and instead of the classical Fourier law, proposed a time-delayed constitutive equation. Expanding the delayed terms into Taylor's series and considering the partial sums of the expansions, Tzou (Tzou, 1995b) obtained approximations of the dual-phase-lag model, which are generalisations of the nonclassical Maxwell-Cattaneo-Vernotte law of heat conduction with one phase-lag parameter (Maxwell, 1867; Cattaneo, 1949; Vernotte, 1958). D. Chandrasekharaiah (Chandrasekharaiah, 1998) constructed a nonclassical model of thermoelastic solids in which the Fourier law of heat conduction is replaced with an approximation of the dual-phase-lag model proposed by Tzou, where the second-order and first-order terms are retained in the Taylor series expansions of the heat flux and temperature gradient, respectively. In the paper (Dreher et al., 2009), it was proved that the delay equation corresponding to the Tzou's law of heat conduction is ill-posed. The initial-boundary value problem corresponding to the Chandrasekharaiah-Tzou three-dimensional model for homogeneous isotropic solids with general mixed boundary conditions was studied in (Avalishvili & Avalishvili, 2013) and the result on existence and uniqueness of weak solution was obtained, and, later on, a particular case of this problem with homogeneous Dirichlet boundary conditions was investigated in (Maes & Van Bockstal, 2021). The uniqueness and continuous dependence results were obtained in the spaces of classical smooth enough functions for the initial-boundary value problem corresponding to the Chandrasekharaiah-Tzou model in (Chiriţă, 2017). The compatibility of various high-order approximations of Tzou's dual-phase-lag heat conduction model with the second law of thermodynamics was studied in (Chiriţă et al., 2017). The uniqueness results in the spaces of classical smooth enough functions for models of thermoelastic solids for three particular high-order approximations of Tzou's dual-phase-lag heat conduction law were obtained in (Zampoli, 2018), and the uniqueness and continuous dependence results for the heat conduction equation with the high-order approximations of Tzou's dual-phase-lag model were proved in (Chiriţă, 2019). Numerous papers are devoted to the study of thermoelastic waves, stability and spatial behaviour of solutions, numerical and analytical methods of solution and related topics for the dual-phase-lag models for thermoelastic solids (see Quintanilla & Racke, 2007; Tzou, 2015; Saidi & Abouelregal, 2021; Shakeriaski et al., 2021; Gupta et al., 2022, and the references given therein).

The present paper is devoted to the investigation of the nonclassical model of thermoelastic solids with two phase-lags, which is a generalisation of the Chandrasekharaiah-Tzou model (Chandrasekharaiah, 1998), where the classical Fourier law of heat conduction is replaced with an approximation of Tzou's dual-phase-lag heat conduction law in which the lagged heat flux and temperature gradient are substituted with high degree partials sums of the corresponding Taylor expansions with respect to the time variable. We consider the linear three-dimensional initial-boundary value problem corresponding to the nonclassical model of thermoelastic solids, with general mixed boundary conditions, where, on certain parts of the boundary, the densities of surface force and heat flux along the outward normal vector are given, and, on the corresponding remaining parts, the displacement and temperature vanish. We obtain integral relations, which are equivalent to the original differential equations in the spaces of smooth enough functions, and, based on the integral relations, we consider the variational formulation in suitable spaces of vector-valued distributions with respect to the time variable with values in the Sobolev spaces of the three-dimensional initial-boundary value problem. We investigate the existence, uniqueness and continuous dependence of the solution on the given data in suitable function spaces.

We denote by $W^{r,2}(D) = H^r(D)$ and $H^{\hat{r}}(\hat{\Gamma})$, $r, \hat{r} \in \mathbf{R}$, $r \geq 0$, $0 \leq \hat{r} \leq 1$, the Sobolev spaces of orders $r, \hat{r}$ based on the spaces $H^0(D) = L^2(D)$ and $H^0(\hat{\Gamma}) = L^2(\hat{\Gamma})$ of square-integrable functions, respectively, where $D \subset \mathbf{R}^p$, $p \in \mathbf{N}$, is a bounded Lipschitz domain (McLean, 2000) and $\hat{\Gamma} \subset \partial D$ is a Lipschitz surface. We denote by $\mathbf{H}^r(D) = [H^r(D)]^3$, $\mathbf{H}^{\hat{r}}(\hat{\Gamma}) = [H^{\hat{r}}(\hat{\Gamma})]^3$, $\mathbf{L}^2(D) = [L^2(D)]^3$, $\mathbf{L}^s(\hat{\Gamma}) = [L^s(\hat{\Gamma})]^3$, $r \geq 0$, $0 \leq \hat{r} \leq 1$, $s \geq 1$, $r, \hat{r}, s \in \mathbf{R}$, the corresponding spaces of vector-valued functions. The trace operators are denoted by $tr_{\hat{\Gamma}} : H^1(D) \to H^{1/2}(\hat{\Gamma})$ and $\mathbf{tr}_{\hat{\Gamma}} : \mathbf{H}^1(D) \to \mathbf{H}^{1/2}(\hat{\Gamma})$. For any $m \in \mathbf{N} \cup \{0\}$, we denote by $C^{m,1}(\bar{D})$ the space of functions whose all partial derivatives up to the order m are Lipschitz continuous on $\bar{D}$ (Ciarlet, 2013). For a Banach space X, we denote by $C([0,T]; X)$ the space of continuous functions on $[0,T]$ with values in X. $L^q(0,T;X)$, $1 \leq q \leq \infty$, is the space of such measurable functions $g : (0,T) \to X$ that $\|g(t)\|_X \in L^q(0,T)$ and the generalized first, second and arbitrary k-th order derivatives of g are denoted by $g' = dg/dt$, $g'' = d^2g/dt^2$ and $g^{(k)} = d^kg/dt^k$, $k \in \mathbf{N} \cup \{0\}$, $g^{(0)} = g$ (Dautray & Lions, 2000).

Let us consider a thermoelastic body with an initial configuration $\bar{\Omega} \subset \mathbf{R}^3$, which consists of an anisotropic homogeneous thermoelastic material. The body is clamped along a part Γ_0 of the Lipschitz boundary $\Gamma = \partial\Omega$ and, on the remaining part $\Gamma_1 = \Gamma \setminus \Gamma_0$, a surface force with density $\mathbf{g} = (g_i) : \Gamma_1 \times (0,T) \to \mathbf{R}^3$ is given, where $\Gamma = \Gamma_0 \cup \Gamma_1$ is a Lipschitz dissection (McLean, 2000) of Γ; the temperature θ vanishes along a part $\Gamma_0^\theta \subset \Gamma$ of the boundary and the heat flux along the outward normal vector of Γ, with density $\tilde{g}^\theta : \Gamma_1^\theta \times (0,T) \to \mathbf{R}$, is given on the remaining part $\Gamma_1^\theta = \Gamma \setminus \Gamma_0^\theta$ of the boundary, where $\Gamma = \Gamma_0^\theta \cup \Gamma_1^\theta$ is a Lipschitz dissection of Γ. The body is subjected to an applied body force with density $\mathbf{f} = (f_i) : \Omega \times (0,T) \to \mathbf{R}^3$ and a heat source with density $\tilde{f}^\theta : \Omega \times (0,T) \to \mathbf{R}$. The nonclassical dynamic linear three-dimensional model of stress-strain state of the thermoelastic body $\bar{\Omega}$, in which the classical Fourier law of heat conduction is replaced with the high-order approximation

$$\sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \frac{\partial^r q_i(x,t)}{\partial t^r} = -\sum_{j=1}^{3} \lambda_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_\theta} \frac{\tau_1^s}{s!} \frac{\partial^s \theta(x,t)}{\partial t^s} \right), \quad i = 1,2,3,$$

of the dual-phase-lag law of heat conduction proposed by Tzou (Tzou, 1995a, 1995b), where $\vec{q} = (q_i)_{i=1}^3$ is the heat flux vector, is given by the following initial-boundary value problem in differential form:

$$\frac{\partial^2 u_i}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\sum_{p,q=1}^{3} \mu_{ijpq} e_{pq}(\mathbf{u}) + \eta_{ij}\theta \right) = f_i \quad \text{in } \Omega \times (0,T), \ i = 1,2,3, \tag{1}$$

$$\chi \sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \frac{\partial^{r+1}\theta}{\partial t^{r+1}} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \left(\lambda_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_\theta} \frac{\tau_1^s}{s!} \frac{\partial^s \theta}{\partial t^s} \right) \right) - \Theta_0 \sum_{i,j=1}^{3} \eta_{ij} e_{ij} \left(\sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \frac{\partial^{r+1}\mathbf{u}}{\partial t^{r+1}} \right) = \sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \frac{\partial^r \tilde{f}^\theta}{\partial t^r} \quad \text{in } \Omega \times (0,T), \tag{2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0 \times (0,T), \qquad \sum_{j=1}^{3} \left(\sum_{p,q=1}^{3} \mu_{ijpq} e_{pq}(\mathbf{u}) + \eta_{ij}\theta \right) \nu_j = g_i \quad \text{on } \Gamma_1 \times (0,T), \ i = 1,2,3, \tag{3}$$

$$\theta = 0 \quad \text{on } \Gamma_0^\theta \times (0,T), \qquad -\sum_{i,j=1}^{3} \lambda_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_\theta} \frac{\tau_1^s}{s!} \frac{\partial^s \theta}{\partial t^s} \right) \nu_i = \sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \frac{\partial^r \tilde{g}^\theta}{\partial t^r} \quad \text{on } \Gamma_1^\theta \times (0,T), \tag{4}$$

$$\begin{aligned} &\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \frac{\partial \mathbf{u}}{\partial t}(x,0) = \mathbf{u}_1(x), \\ &\theta(x,0) = \theta_0(x), \quad \frac{\partial \theta}{\partial t}(x,0) = \theta_1(x), \quad . \ . \ . \ . \ . \ , \quad \frac{\partial^{n_q}\theta}{\partial t^{n_q}}(x,0) = \theta_{n_q}(x), \end{aligned} \qquad x \in \Omega, \tag{5}$$

where $e_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$, $i,j = 1,2,3$, $\mathbf{v} = (v_i)_{i=1}^3 : \Omega \to \mathbf{R}^3$, is the linearized strain tensor, $\boldsymbol{\nu} = (\nu_i)_{i=1}^3$ is the unit outward normal to Γ, $\mathbf{u} = (u_i)_{i=1}^3 : \Omega \times (0,T) \to \mathbf{R}^3$ is the displacement vector-function of the thermoelastic body, $\theta : \Omega \times (0,T) \to \mathbf{R}$ is the temperature distribution, $\mathbf{u}_0 = (u_{0i})_{i=1}^3$ and $\mathbf{u}_1 = (u_{1i})_{i=1}^3$ are the initial displacement and velocity vector-functions, and θ_0 is the initial distribution of temperature, θ_1 is the rate of change of temperature, and θ_k, $k = 2,\dots,n_q$, are the k-th order derivatives of temperature with respect to the time variable at the initial moment, ρ is the mass density in the reference configuration, $(\mu_{ijpq})_{i,j,p,q=1}^3$ is the elasticity tensor, $(\eta_{ij})_{i,j=1}^3$ is the stress-temperature tensor, $\chi > 0$ is the volumetric heat capacity, $(\lambda_{ij})_{i,j=1}^3$ is the thermal conductivity tensor, $\Theta_0 > 0$ is the temperature of the thermoelastic body in the natural state of no deformation, which is considered as a reference temperature, $\tau_0 \geq 0$ and $\tau_1 \geq 0$

stand for the heat flux and temperature gradient phase-lags, respectively. Note that the considered model is a generalization of the Chandrasekharaiah-Tzou dual-phase-lag model and coincides with it in the case of $n_q = 2$ and $n_\theta = 1$, the Lord-Shulman model (Lord & Shulman, 1967; Avalishvili & Avalishvili, 2014) is a particular case of this model for $n_q = 1$ and $n_\theta = 0$, and in the case of $\tau_0 = \tau_1 = 0$ the nonclassical model (1)-(5) reduces to the classical linear three-dimensional model of thermoelastic solids.

We assume that the elasticity tensor $(\mu_{ijpq})_{i,j,p,q=1}^3$, the stress-temperature tensor $(\eta_{ij})_{i,j=1}^3$ and the thermal conductivity tensor $(\lambda_{ij})_{i,j=1}^3$ satisfy the following symmetry conditions:

$$\mu_{ijpq} = \mu_{ijqp} = \mu_{jipq}, \quad \eta_{ij} = \eta_{ji}, \quad \lambda_{ij} = \lambda_{ji}, \qquad i,j,p,q = 1,2,3. \tag{6}$$

If $\mathbf{u} = (u_i)_{i=1}^3$ and θ are smooth enough, then by multiplying equation (1) by arbitrary continuously differentiable functions $v_i : \bar{\Omega} \to \mathbf{R}$ $(i = 1,2,3)$, which vanish on Γ_0, and equation (2) by a continuously differentiable function $\varphi : \bar{\Omega} \to \mathbf{R}$, such that $\varphi = 0$ on Γ_0^θ, by integrating on Ω, using the Green identity (McLean, 2000), and taking into account the symmetry properties (6) and the boundary conditions (3), (4), we obtain the following integral relations:

$$\sum_{i=1}^{3} \int_\Omega \rho \frac{\partial^2 u_i}{\partial t^2} v_i dx + \sum_{i,j,p,q=1}^{3} \int_\Omega \mu_{ijpq} e_{pq}(\mathbf{u}) e_{ij}(\mathbf{v}) dx + \sum_{i,j=1}^{3} \int_\Omega \eta_{ij} \theta e_{ij}(\mathbf{v}) dx = \sum_{i=1}^{3} \int_\Omega f_i v_i dx + \sum_{i=1}^{3} \int_{\Gamma_1} g_i v_i d\Gamma, \tag{7}$$

$$\int_\Omega \chi \sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \frac{\partial^{r+1}\theta}{\partial t^{r+1}} \varphi dx + \sum_{i,j=1}^{3} \int_\Omega \lambda_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_\theta} \frac{\tau_1^s}{s!} \frac{\partial^s \theta}{\partial t^s} \right) \frac{\partial \varphi}{\partial x_i} dx -$$

$$- \Theta_0 \sum_{i,j=1}^{3} \int_\Omega \eta_{ij} e_{ij} \left(\sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \frac{\partial^{r+1}\mathbf{u}}{\partial t^{r+1}} \right) \varphi dx = \int_\Omega f^\theta \varphi dx - \int_{\Gamma_1^\theta} g^\theta \varphi d\Gamma, \tag{8}$$

where $f^\theta = \sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \frac{\partial^r \tilde{f}^\theta}{\partial t^r}$, $g^\theta = \sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \frac{\partial^r \tilde{g}^\theta}{\partial t^r}$. Therefore, if $\mathbf{u} = (u_i)_{i=1}^3$ and θ are solutions of equations (1), (2) and satisfy boundary conditions (3), (4), then $\mathbf{u} = (u_i)_{i=1}^3$ and θ are solutions of (7), (8). Conversely, if $\mathbf{u} = (u_i)_{i=1}^3$ and θ are smooth enough solutions of equations (7), (8), then by using the Green identity and density arguments, we obtain that $\mathbf{u} = (u_i)_{i=1}^3$ and θ satisfy equations (1), (2) and boundary conditions (3), (4). So, the initial-boundary value problem (1)-(5) corresponding to the nonclassical dynamic three-dimensional model of thermoelastic solids with two phase-lags is equivalent to equations (7), (8), together with initial conditions (5) in the spaces of smooth enough functions.

We identify the unknown vector-function $\mathbf{u}$ and the function θ with vector-functions defined on $[0,T]$ with values in suitable spaces of functions defined on Ω, and by applying the integral relations (7), (8) in the case of $n_q = n_\theta + 1$ we obtain the following variational formulation of the initial-boundary value problem (1)-(5) in the spaces of vector-valued distributions: Find the unknown vector-function $\mathbf{u}, \mathbf{u}', \ldots, \mathbf{u}^{(n_q)} \in C([0,T]; \mathbf{V}(\Omega))$, $\mathbf{u}^{(n_q+1)} \in L^\infty(0,T; \mathbf{V}(\Omega))$, $\mathbf{u}^{(n_q+2)} \in L^\infty(0,T; \mathbf{L}^2(\Omega))$, $\theta, \theta', \ldots, \theta^{(n_q-1)} \in C([0,T]; V^\theta(\Omega))$, $\theta^{(n_q)} \in L^\infty(0,T; V^\theta(\Omega))$, $\theta^{(n_q+1)} \in L^\infty(0,T; L^2(\Omega))$, which satisfy the following equations in the sense of distributions on $(0,T)$,

$$(\rho \mathbf{u}'', \mathbf{v})_{\mathbf{L}^2(\Omega)} + a(\mathbf{u}, \mathbf{v}) + b(\theta, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2(\Omega)} + (\mathbf{g}, \mathbf{tr}_{\Gamma_1}(\mathbf{v}))_{\mathbf{L}^2(\Gamma_1)}, \qquad \forall \mathbf{v} \in \mathbf{V}(\Omega), \tag{9}$$

$$\left(\chi \sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \theta^{(r+1)}, \varphi \right)_{L^2(\Omega)} + a^\theta \left(\sum_{s=0}^{n_\theta} \frac{\tau_1^s}{s!} \theta^{(s)}, \varphi \right) - \Theta_0 b^\theta \left(\sum_{r=0}^{n_q} \frac{\tau_0^r}{r!} \mathbf{u}^{(r+1)}, \varphi \right) =$$

$$=(f^{\theta},\varphi)_{L^2(\Omega)}-(g^{\theta},tr_{\Gamma_1^{\theta}}(\varphi))_{L^2(\Gamma_1^{\theta})},\quad \forall\varphi\in V^{\theta}(\Omega), \tag{10}$$

together with the initial conditions

$$\mathbf{u}(0)=\mathbf{u}_0,\ \mathbf{u}'(0)=\mathbf{u}_1,\ \theta(0)=\theta_0,\ \theta'(0)=\theta_1,\ \ .\ \ .\ \ .\ \ .\ ,\theta^{(n_q)}(0)=\theta_{n_q}, \tag{11}$$

where $\mathbf{V}(\Omega)=\{\mathbf{v}\in\mathbf{H}^1(\Omega);\mathbf{tr}_{\Gamma}(\mathbf{v})=\mathbf{0}\ \text{ on }\Gamma_0\}$, $V^{\theta}(\Omega)=\{\varphi\in H^1(\Omega);tr_{\Gamma}(\varphi)=0\ \text{ on }\Gamma_0^{\theta}\}$,

$$a(\tilde{\mathbf{v}},\mathbf{v})=\int_{\Omega}\sum_{i,j,p,q=1}^{3}\mu_{ijpq}e_{pq}(\tilde{\mathbf{v}})e_{ij}(\mathbf{v})dx,\quad \forall\mathbf{v},\tilde{\mathbf{v}}\in\mathbf{H}^1(\Omega),\qquad a^{\theta}(\tilde{\varphi},\varphi)=\int_{\Omega}\sum_{i,j=1}^{3}\lambda_{ij}\frac{\partial\tilde{\varphi}}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dx,\quad \forall\varphi,\tilde{\varphi}\in H^1(\Omega),$$

$$b(\varphi,\mathbf{v})=b^{\theta}(\mathbf{v},\varphi)=\int_{\Omega}\varphi\sum_{i,j=1}^{3}\eta_{ij}e_{ij}(\mathbf{v})dx,\quad \forall\varphi\in L^2(\Omega),\mathbf{v}\in\mathbf{H}^1(\Omega),$$

$(.,.)_{\mathbf{L}^2(\Omega)}$, $(.,.)_{L^2(\Omega)}$, $(.,.)_{\mathbf{L}^2(\Gamma_1)}$ and $(.,.)_{L^2(\Gamma_1^{\theta})}$ are scalar products in the spaces $\mathbf{L}^2(\Omega)$, $L^2(\Omega)$, $\mathbf{L}^2(\Gamma_1)$ and $L^2(\Gamma_1^{\theta})$, respectively.

For problem (9)-(11), which is equivalent to the initial-boundary value problem (1)-(5) in the spaces of smooth enough functions, the following theorem is valid.

Theorem. Suppose that Ω is a bounded Lipschitz domain, $n_q=n_{\theta}+1$, $\rho\in C^{n_q-1,1}(\overline{\Omega})$, $\mu_{ijpq}\in C^{n_q,1}(\overline{\Omega})$, $\eta_{ij}\in C^{n_q,1}(\overline{\Omega})$, $\lambda_{ij}\in C^{0,1}(\overline{\Omega})$, $\chi\in L^{\infty}(\Omega)$, $i,j,p,q=1,2,3$, $\rho(x)>c_{\rho}=const>0$, $\chi(x)>c_{\chi}=const>0$, for almost all $x\in\Omega$, and the symmetry conditions (6) and the following positive definiteness conditions are valid

$$\sum_{i,j,p,q=1}^{3}\mu_{ijpq}(x)\varepsilon_{pq}\varepsilon_{ij}\ge c_{\mu}\sum_{i,j=1}^{3}(\varepsilon_{ij})^2,\quad \forall\varepsilon_{ij}\in\mathbf{R},\ \varepsilon_{ij}=\varepsilon_{ji},\ i,j=1,2,3,\ c_{\mu}=const>0,\ x\in\Omega,$$

$$\sum_{p,q=1}^{3}\lambda_{pq}(x)\varepsilon_p\varepsilon_q\ge c_{\lambda}\sum_{p=1}^{3}(\varepsilon_p)^2,\quad \forall\varepsilon_p\in\mathbf{R},\ p=1,2,3,\ c_{\lambda}=const>0,\ x\in\Omega,$$

and $\tau_0>0$, $\tau_1>0$. If $\mathbf{f}^{(k)}\in C([0,T];\mathbf{H}^{n_q-k}(\Omega))$, $k=0,...,n_q-1$, $\mathbf{f}^{(n_q)},\mathbf{f}^{(n_q+1)}\in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g}^{(\bar{k})}\in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $\bar{k}=0,...,n_q+2$, $f^{\theta},f^{\theta'}\in L^2(0,T;L^2(\Omega))$, $g^{\theta},g^{\theta'},g^{\theta''}\in L^2(0,T;L^{4/3}(\Gamma_1^{\theta}))$, and the initial conditions $\mathbf{u}_0\in\mathbf{H}^{n_q+2}(\Omega)\cap\mathbf{V}(\Omega)$, $\mathbf{u}_1\in\mathbf{H}^{n_q+1}(\Omega)\cap\mathbf{V}(\Omega)$, $\theta_{\tilde{k}}\in H^{n_q+1-\tilde{k}}(\Omega)\cap V^{\theta}(\Omega)$, $\tilde{k}=0,...,n_q$, satisfy the following compatibility conditions

$$g_i^{(k)}(0)=tr_{\Gamma_1}\left(\sum_{j=1}^{3}\left(\sum_{p,q=1}^{3}\mu_{ijpq}e_{pq}(\mathbf{u}_k)+\eta_{ij}\theta_k\right)\nu_j\right),\quad i=1,2,3,\ k=0,1,2,...,n_q,$$

$$g^{\theta}(0)=-tr_{\Gamma_1^{\theta}}\left(\sum_{i,j=1}^{3}\lambda_{ij}\frac{\partial}{\partial x_j}\left(\sum_{s=0}^{n_{\theta}}\frac{\tau_1^s}{s!}\theta_s\right)\nu_j\right),$$

where $\mathbf{v}=(\nu_i)_{i=1}^3$ is the unit outward normal vector to Γ,

$$u_{ki}=\frac{1}{\rho}\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\sum_{p,q=1}^{3}\mu_{ijpq}e_{pq}(\mathbf{u}_{k-2})+\eta_{ij}\theta_{k-2}\right)+f_i^{(k-2)}(0)\right),\quad i=1,2,3,\ k=2,3,...,n_q,$$

then the initial-boundary value problem (9)-(11) possesses a unique solution, which continuously depends on the given data, i.e. the mapping

$$(\mathbf{u}_0,\mathbf{u}_1,(\theta_k)_{k=0}^{n_q},(\mathbf{f}^{(\bar{k})})_{\bar{k}=0}^{n_q-1},\mathbf{f}^{(n_q)},(\mathbf{g}^{(\tilde{k})})_{\tilde{k}=0}^{n_q+1},f^{\theta},g^{\theta},g^{\theta'})\to((\mathbf{u}^{(k)})_{k=0}^{n_q},\mathbf{u}^{(n_q+1)},(\theta^{(\bar{k})})_{\bar{k}=0}^{n_q-1},\theta^{(n_q)})$$

is linear and continuous from space

$$\mathbf{H}^{n_q+1}(\Omega)\times\mathbf{H}^{n_q}(\Omega)\times\prod_{k=0}^{n_q}H^{n_q-k}(\Omega)\times\prod_{\bar{k}=0}^{n_q-1}C([0,T];\mathbf{H}^{n_q-1-\bar{k}}(\Omega))\times L^2(0,T;\mathbf{L}^2(\Omega))\times$$
$$\times\left(L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))\right)^{n_q+2}\times L^2(0,T;L^2(\Omega))\times L^2(0,T;L^{4/3}(\Gamma_1^\theta))\times L^2(0,T;L^{4/3}(\Gamma_1^\theta))$$

to space

$$\left(C([0,T];\mathbf{V}(\Omega))\right)^{n_q+1}\times C([0,T];\mathbf{L}^2(\Omega))\times\left(C([0,T];V^\theta(\Omega))\right)^{n_q}\times C([0,T];L^2(\Omega)),$$

and the following energy equality is valid:

$$E(t)=E(0)+L(\mathbf{u},\theta,\mathbf{f},f^\theta,\mathbf{g},g^\theta)(t),\qquad \forall t\in[0,T],$$

where

$$E(t)=\frac{1}{2}\left(\rho\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^{r+1}\mathbf{u}(t)}{\partial t^{r+1}},\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^{r+1}\mathbf{u}(t)}{\partial t^{r+1}}\right)_{\mathbf{L}^2(\Omega)}+\frac{1}{2}a\left(\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\mathbf{u}(t)}{\partial t^r},\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\mathbf{u}(t)}{\partial t^r}\right)+$$
$$+\frac{1}{2\Theta_0}\left(\chi\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\theta(t)}{\partial t^r},\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\theta(t)}{\partial t^r}\right)_{L^2(\Omega)}+\frac{1}{\Theta_0}\int_0^t\sum_{s=0}^{n_q-1}\frac{(\tau_0\tau_1)^s}{(s!)^2}a^\theta(\theta^{(s)}(\tau),\theta^{(s)}(\tau))d\tau+$$
$$+\frac{1}{2\Theta_0}\sum_{s=0}^{n_q-1}\frac{\tau_0^{s+1}\tau_1^s}{(s+1)!s!}a^\theta(\theta^{(s)}(t),\theta^{(s)}(t))+\frac{1}{2\Theta_0}\sum_{s=1}^{n_q-1}\frac{\tau_0^{s-1}\tau_1^s}{(s-1)!s!}a^\theta(\theta^{(s-1)}(t),\theta^{(s-1)}(t))+$$
$$+\frac{1}{\Theta_0}\int_0^t\sum_{\substack{r=0\\ r\neq s-1,s,s+1}}^{n_q-1}\sum_{s=0}^{n_q-1}\frac{\tau_0^r\tau_1^s}{r!s!}a^\theta(\theta^{(s)}(\tau),\theta^{(r)}(\tau))d\tau+\frac{1}{\Theta_0}\sum_{s=0}^{n_q-2}\frac{\tau_0^{n_q}\tau_1^s}{n_q!s!}a^\theta(\theta^{(s)}(t),\theta^{(n_q-1)}(t))-$$
$$-\frac{1}{\Theta_0}\int_0^t\sum_{s=0}^{n_q-2}\frac{\tau_0^{n_q}\tau_1^s}{n_q!s!}a^\theta(\theta^{(s+1)}(\tau),\theta^{(n_q-1)}(\tau))d\tau,$$

$$L(\mathbf{u},\theta,\mathbf{f},f^\theta,\mathbf{g},g^\theta)(t)=\int_0^t\left(\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\mathbf{f}(\tau)}{\partial t^r},\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^{r+1}\mathbf{u}(\tau)}{\partial t^{r+1}}\right)_{\mathbf{L}^2(\Omega)}d\tau+$$
$$+\left(\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\mathbf{g}(t)}{\partial t^r},\mathbf{tr}_{\Gamma_1}\left(\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\mathbf{u}(t)}{\partial t^r}\right)\right)_{\mathbf{L}^2(\Gamma_1)}-\left(\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\mathbf{g}(0)}{\partial t^r},\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\mathbf{u}(0)}{\partial t^r}\right)_{\mathbf{L}^2(\Gamma_1)}-$$
$$-\int_0^t\left(\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^{r+1}\mathbf{g}(\tau)}{\partial t^{r+1}},\mathbf{tr}_{\Gamma_1}\left(\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\mathbf{u}(\tau)}{\partial t^r}\right)\right)_{\mathbf{L}^2(\Gamma_1)}d\tau+\frac{1}{\Theta_0}\int_0^t\left(f^\theta(\tau),\sum_{r=0}^{n_q}\frac{\tau_0^r}{r!}\frac{\partial^r\theta(\tau)}{\partial t^r}\right)_{L^2(\Omega)}d\tau-$$
$$-\frac{1}{\Theta_0}\int_0^t\left(\sigma^\theta(\tau),tr_{\Gamma_1^\theta}\left(\sum_{r=0}^{n_q-1}\frac{\tau_0^r}{r!}\theta^{(r)}(\tau)\right)\right)_{L^2(\Gamma_1^\theta)}d\tau-\frac{1}{\Theta_0}\left(g^\theta(t),tr_{\Gamma_1^\theta}\left(\frac{\tau_0^{n_q}}{n_q!}\theta^{(n_q-1)}(t)\right)\right)_{L^2(\Gamma_1^\theta)}+$$
$$+\frac{1}{\Theta_0}\left(g^\theta(0),tr_{\Gamma_1^\theta}\left(\frac{\tau_0^{n_q}}{n_q!}\theta^{(n_q-1)}(0)\right)\right)_{L^2(\Gamma_1^\theta)}+\frac{1}{\Theta_0}\int_0^t\left(g^{\theta\prime}(\tau),tr_{\Gamma_1^\theta}\left(\frac{\tau_0^{n_q}}{n_q!}\theta^{(n_q-1)}(\tau)\right)\right)_{L^2(\Gamma_1^\theta)}d\tau.$$

მათემატიკა

თერმოდრეკადი სხეულების ორფაზიანი დაგვიანებით დროით მაღალი რიგის დიფერენციალური მოდელების შესახებ

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ნაშრომში განხილულია თერმოდრეკადი სხეულების ორფაზიანი დაგვიანებით დინამიკური სამგანზომილებიანი მოდელის დროითი ცვლადის მიმართ მაღალი რიგის დიფერენციალური აპროქსიმაცია. გამოკვლეულია ორი ფაზის წანაცვლების პარამეტრზე დამოკიდებული არაკლასიკური მოდელის შესაბამისი საწყის-სასაზღვრო ამოცანა შერეული სასაზღვრო პირობებით, სადაც საზღვრის გარკვეულ ნაწილებზე გადაადგილება და ტემპერატურა ნულის ტოლია, ხოლო საზღვრის შესაბამის დარჩენილ ნაწილებზე მოცემულია ზედაპირული ძალის და გარე ნორმალის გასწვრივ სითბოს ნაკადის სიმკვრივეები. მიღებულია საწყის-სასაზღვრო ამოცანის ვარიაციული ფორმულირება და სათანადო ვექტორული მნიშვნელობების მქონე განაწილებების სივრცეებში დამტკიცებულია არსებობისა და ერთადერთობის შედეგი და ამონახსნის უწყვეტად დამოკიდებულება მონაცემებზე.

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Received June, 2025