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Physics

Two-Dimensional Theoretical Model of the Field with a Periodic Configuration

Avtandil Shurgaia

Razmadze Mathematical Institute, Ivane Javakhishvili Tbilisi State University, Georgia

(Presented by Academy Member Aleksandre Kvinikhidze)

Abstract. The quantum properties of a (1+1)-dimensional scalar theory on a cylinder with a compact spatial part are considered. A simple model with a quartic potential is examined, which features two distinct vacua separated by a potential barrier. The classical equations allow for exact finite-energy periodic field configurations. These field equations are explicitly solved in terms of the Jacobian elliptic function, which accounts for the periodicity of the solution. Such field configurations are of interest due to their relevance in phase transitions from classical thermal processes to quantum tunneling depending on the energy of the periodic instanton. Explicitly finding similar field configurations in real space-time is challenging; thus, simplified models hold significant interest. Additionally, the temporal fluctuations of the obtained field configurations are analyzed using reasonable approximations. The eigenvalues of the corresponding equations are discrete, and a negative eigenvalue is present among them. An explicit expression for this negative eigenvalue enables the calculation of the lifetime of such a quantum state. The collective coordinates method is applied to the Schrödinger equation to develop a perturbation theory in inverse powers of the coupling constant while maintaining manifest translational invariance explicitly at every order of the perturbation series. In the quadratic approximation for quantum fluctuations, the S-matrix is calculated within the holomorphic representation of the Feynman path integrals. *© 2025 Bull. Georg. Natl. Acad. Sci.*

Keywords: periodic instantons, collective coordinates method, lifetime

Introduction

Periodic field configurations, known as periodic instantons, have been of interest to researchers for many years due to their important role in the process of quantum tunneling and the associated phase transitions. The attractiveness of such field objects lies in their ability to interpolate between the vacuum and saddle point states (sphalerons), depending on the total energy, and thus have the potential to be responsible for the phase transition from classical processes at high energies to quantum tunneling ones at low energies. Along with stable field configurations of finite energy more recently unstable configurations like bounces (Zichichi, 1979), sphalerons (Manton, 1983; Klinkhamer & Manton, 1984) and periodic instantons (Manton & Samols, 1988; Liang et al.,1992) have been investigated. The complexity of actual physical theories does not allow for obtaining exact results. From this point of view, (1+1)-dimensional scalar theory is subjected to intense study, specifically the models on a compact space (Pawellek, 2009; Pawellek, 2009). In the present paper, we are interested in periodic classical field configurations and their quantum properties in two-dimensional scalar theory, the spatial part of which is a circle with the circumference L. We use the symmetry properties of the theory and construct a perturbation theory using the collective coordinates method (Shurgaia et al., 2007 and references therein) around the classical field configuration.

Model. Thus, we will study the quantum behaviour of the system with the classical Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi(x,t)\partial^\mu\varphi(x,t) - U(\varphi(x,t),g) \tag{1}$$

with the potential to satisfy the condition $U(\varphi(x,t),g) = \dfrac{1}{g^2}U(g\varphi(x,t),1)$. The system is placed in space-time with the spatial coordinate $0 \le x \le L$. The corresponding quantum Hamiltonian is

$$H = \int_0^L dx\left\{\frac{1}{2}\pi^2(x) + \frac{1}{2}\varphi(x)\left(\frac{\partial\varphi(x)}{\partial x}\right)^2 + U(\varphi(x),g)\right\}. \tag{2}$$

We will study the Schrödinger equation

$$(H - E)\Psi(\varphi(x),\ \pi(x) = 0 \tag{3}$$

subject to the following commutation relation: $[\varphi(x),\ \pi(y) = i\delta(x-y)$. The evolution of the system is subject to the conservation of momentum

$$P = -\int_0^L dx\frac{\partial\varphi(x)}{\partial x}\ \pi(x)\cdot \tag{4}$$

Collective coordinates. Following the collective coordinates method, we introduce the following transformation of the field $\varphi(x)$ (with $\varphi_0(x)$ to be a c-number):

$$\varphi(x) = g\varphi_0(x-q) + \Phi(x-q)\ . \tag{5}$$

The parameter q, being called a collective coordinate, together with the new field $\Phi(x)$ forms now a new set of variables (operators) for the system: $\{\varphi(x),\ \pi(x)\} \to \{q, p, \Phi(x), \Pi(x)\}$. Thereby, the phase space has been expanded, and in order to keep the number of independent variables unchanged, an additional condition must be imposed (actually, this is a gauge condition imposed in the new enlarged phase space):

$$\int_0^L dx N(x)\Phi(x) = 0, \tag{6}$$

where $N(x)$ is an arbitrary function. We need one more additional condition, but as will be shown below, this condition appears as a constraint in the extended space. One can always choose $N(x)$ such that the equality

$$\int_0^L dx N(x)\frac{\partial\varphi_0(x)}{\partial x} = 1 \tag{7}$$

holds. The operator $\pi(x)$ is considered as a functional derivative $-i\dfrac{\delta}{\delta\varphi(x)}$ in a Hilbert space spanned by the vectors $|\psi>$ and hence, should be expressed through the set of new variables. For this, we need to define the projection operator $A(x, y) = \delta(x-y) - \dfrac{\partial\varphi_0(x)}{\partial x}N(y)$ with the properties

$$\int_0^L dxA(x,\, y)\,\frac{\partial\varphi_0(y)}{\partial y} = \int_0^L dN(x)\,A(x,\, y) = 0.$$

Now, it is easy to show that the operator $p(x)$ can be rewritten as follows:

$$\pi(x) = \Pi(x) - \frac{N(x)}{g\left(1+\frac{1}{g}F\right)}\left\{p_q + M(\Phi(x),\Pi(x)\right\}, \tag{8}$$

in which the quantities F and $M(\Phi(x),\Pi(x))$ are defined as

$$F = \int_0^L dxN(x)\frac{\partial\Phi(x)}{\partial x}, \qquad M(\Phi(x),\Pi(x) = \int_0^L dx\frac{\partial\Phi(x)}{\partial x}\Pi(x). \tag{9}$$

Besides, the operator $\Pi(x)$, determined as

$$\Pi(x) = \int_0^L dyA(y,\, x)(-i\,\delta/(\delta\varphi(x))$$

satisfies the condition

$$\int_0^L dx\Pi(x)\frac{\partial\varphi_0(x)}{\partial x} = 0. \tag{10}$$

This is the constraint of the theory (we have mentioned above) to be considered as a system with constraints. The operators q and p each conjugate to each other such that $[q,\, p] = i$. Thus, all extra degrees of freedom are now fixed. The operators $\Phi(x)$ and $\Pi(x)$ obey the following commutation relation:

$$[\Phi(x),\; \Pi(x)] = iAx,y). \tag{11}$$

It can be shown that the operator of total momentum of the system reduces to the operator p_q or $P = p_q$. Expressing the Hamiltonian in terms of new set of canonical variables we will see that it depends on q but by changing the integration variable $x \rightarrow x - q$ the integration interval changes from $-q$ to $L - q$. Based on the Leibnitz rule it can be shown that $\frac{dH}{dq} = 0$, which means that H does not depend explicitly on q. This allows us to factorize the wave function $\Psi(\Phi(x),q)$ as $\Psi(\Phi(x),q) = \exp(ig^2Iq)\Psi'(\Phi(x))$. Now we need to transform the wave function $\Psi'(\Phi(x))$, indeed

$$\Psi'(\Phi(x)) = \exp\left(ig\int_0^L dxs(x)\Phi(x)\right)\Psi''(\Phi(x). \tag{12}$$

No, we can make replacements in the Hamiltonian $p_q \rightarrow g^2I,\; \Pi(x) \rightarrow gs(x) + \Pi(x)$. The Hamiltonian will now act on the wave function $\Psi''(\Phi(x)$.

Perturbation series. We have not used any approximation so far, hence the momentum conservation law is taken explicitly into account. At the same time, the Hamiltonian can be expanded in a series of inverse powers of the constant starting with g^2:

$$H = g^2H_0 + gH_1 + H_2 + g^{-1}H_3 + \ldots \tag{13}$$

Expanding the energy E and the wave function $\Psi''(\Phi(x)$ in similar series

$$E = g^2E_0 + gE_1 + E_2 + g^{-1}E_3 + \ldots,\;\; \Psi''(\Phi(\mathrm{x})) = \Psi_0 + g^{-1}\Psi_1 + g^{-2} + \Psi_2 + \ldots \tag{14}$$

the perturbation theory can be applied to solve the Schrödinger equation as the following system of equations:

$$\left(H_0 - E_0\right)\Psi_0 = 0, \tag{15}$$

$$\left(H_0 - E_0\right)\Psi_1 + \left(H_1 - E_1\right)\Psi_0 = 0, \tag{16}$$

$$\left(H_0 - E_0\right)\Psi_2 + \left(H_1 - E_1\right)\Psi_0 + \left(H_2 - E_2\right)\Psi_0 = 0, \tag{17}$$

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The leading term of the Hamiltonian does not contain field operators and therefore, the corresponding equation $\left(H_0 - E_0\right)\Psi_0 = 0$ is identically obeyed if

$$E_0 = H_0 = \int_0^L dx \left\{ \frac{1}{2}\left(\frac{\partial \varphi_0(x)}{\partial x}\right)^2 + U(\Phi(x),1) + \frac{1}{2}(s(x) - IN(x))^2 \right\}. \tag{18}$$

The next approximation of the system of equations we consider is

$$\left(H_1 - E_1\right)\Psi_0 = 0. \tag{19}$$

which is linear in field operators. The regularity of the function Ψ_0 requires H_1, E_1 to be identically zero. This is achieved by if we choose $s(x)$ such that

$$s(x) - IN(x) = -v\frac{\partial \varphi_0(x)}{\partial x}. \tag{20}$$

and the equation

$$-\left(1 - v^2\right)\frac{\partial^2 \varphi_0(x)}{\partial x^2} + U'\left(\varphi_0(x),1\right) = 0 \tag{21}$$

holds. This is a pure classical equation of motion, the solution of which under periodic boundary conditions for the potential $U(\varphi(x),g)$ of interest is Jacobian elliptic function:

$$\varphi_0(x) = \sqrt{\frac{2k^2}{1+k^2}}\,\mathbf{sn}\left(\sqrt{\frac{2}{1+k^2}}\frac{\mu x}{\sqrt{1-v^2}}, k\right), \tag{22}$$

in which $0 \le k \le 1$ is the modulus of elliptic integral. The solution is periodic with the period

$$L = 4nK(k)\sqrt{\frac{1+k^2}{2}}\frac{\sqrt{1-v^2}}{\mu} \tag{23}$$

with n integer. There are critical values of the quantity L, at which the bifurcations of the stationary points of the energy E_0 occur. These solutions interpolate between the stable field configurations as k^2 tends to 1 (with L approaching the ∞) and the unstable ones (at the top of the potential barrier) for $k^2 \to 0$ latter being called sphalerons (Manton & Samols, 1988; Liang et al., 1992).

The energy E_0e and the momentum I take now the form:

$$E_0 = \frac{n\sqrt{2}\mu[8\left(1+k^2\right)E(k)\left[1-k^2\right)\left(5+k^2\right) + 3\left(1-k^2\right)v^2}{3\left(1+k^2\right)^{3/2}\sqrt{\left(1-v^2\right)}} \tag{24}$$

$$I = \frac{4n\sqrt{2}\mu[\left(1-k^2\right)K(k) + E(k)}{3(1+k^2)^{3/2}\sqrt{1-v^2}} \quad . \tag{25}$$

The expression for the energy E_0 is consistent with that obtained in (Manton & Samols, 1988). It can be verified that v is the center of mass speed. In equations (22)-(25) $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind respectively.

Let us now proceed with the study of quantum correction to the ground state energy, for which the equation

$$\left(H_2 - E_2\right)\Psi_0 = 0 \tag{26}$$

has to be solved. It's quadratic in the field operators $\Phi(x)$ and $\Pi(x)$ and can therefore be diagonalized and reduced to an infinite set of oscillators. The only problem is to take into consideration the constraints that are imposed on $\Phi(x)$ and $\Pi(x)$. We will work in the center of mass system, i.e. where $v = 0$. This simplifies the expression of H_2, in particular

$$H_2 = \int dx \left\{ \frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial \Phi(x)}{\partial x}\right)^2 + \mu^2(3\varphi_0^2(x) - 1)\Phi^2(x) \right\}. \tag{27}$$

We now consider the expansion of operators $\Phi(x)$ and $\Pi(x)$ in terms of functions $V_n(x)$ which are solutions of the equation (taking into account explicitly potential $U(\varphi_0, 1)$)

$$\left\{ -\frac{d^2}{dx^2} + \mu^2\left(\frac{6k^2}{1+k^2}\mathbf{sn}\left(\sqrt{\frac{2}{1+k^2}}, k\mu x\right) - 1\right)\right\} V_n(x) = \mathscr{E}_n V_n(x), \tag{28}$$

$$\Phi(x) = \sum_n{}' \sqrt{\frac{1}{2\mathscr{E}_n}} \left\{ a_n V_n(x) + a_n^+ V_n^*(x) \right\}, \quad \Pi(x) = \mathrm{i} \sum_n{}' \sqrt{\frac{\mathscr{E}_n}{2}} \left\{ a_n^+ V_n^*(x) + a_n V_n(x) \right\}. \tag{29}$$

in which the prime denotes that the sum does not contain the modes with zero energy (translational mode). This system of functions fulfills the constraint

$$\int dx N(x) V_n(x) = 0 \tag{30}$$

and the condition of completeness:

$$\sum \left(V_n(x) V_n^*(y) + V_n(y) V_n^*(x) \right) = A(x, y) \tag{31}$$

which indicates that this system of functions does not contain the function corresponding to the zero mode. Here the creation and annihilation operators obey the commutation relation $[a_n, a_m^+] = \delta_{mn}$ with all other commutators to be zero.

By introducing the notations

$$z = \sqrt{\frac{2}{1+k^2}}\mu x, \quad \lambda = \frac{\mathscr{E}_n^2 + \mu^2}{\sqrt{2}}\sqrt{1+k^2}. \tag{32}$$

the equation (28) can be reduced to the Lame equation:

$$\frac{d^2 V_n(z)}{dz^2} + [\lambda + N(N+1)k^2 \mathbf{sn}(\mathrm{z,k}) V_n(z) = 0. \tag{33}$$

In our case of double-well potential $N = 2$. Periodic solutions of the Lame equation are well studied in (Arscott, 1964). All the eigenvalues of the Lame equation are discrete, and corresponding solutions are known as Lame polynomials. There are in general $2N + 1$ discrete eigenvalues for given N and 5 in our case, one of them is zero, which is excluded from the set of functions $V_n(x)$. This is also confirmed by the completeness condition of the functions $V_n(x)$. It is important, that among these eigenvalues only one is negative (let it denote by $\mathscr{E}^2_{-1}$), indeed $\mathscr{E}^2_{-1} = 2\mu^2(1 - 2\frac{\sqrt{1-k^2(1-k^2)}}{1+k^2} < 0.$ All other eigenvalues are positive. By substituting the expansions (29) into (27), we obtain the diagonal form of H_2:

$$H_2 = \frac{1}{2}\sum_n{}' \mathscr{E}_n(a_n a_n^+ + a_n^+ a_n). \tag{34}$$

The energy of the lowest state (which we name quantum periodic instanton) $E_2 = \frac{1}{2}\sum_n{}' \mathscr{E}_n$ diverges but this is beyond the scope of our interest, since we want to estimate the lifetime of physical system (one can find the details of regularization in Pawelek 2009). Obviously, the energy of the system is a complex quantity. The imaginary part of the energy $\Im E = \frac{\Gamma}{2}$ (with Γ being a decay width) is a measure for the lifetime t_l of the system

$$t_l = \frac{2}{\Gamma} = \frac{\sqrt{2}}{\mu\sqrt{\frac{2\sqrt{(1-\kappa^2(1-\kappa^2)}}{1+\kappa^2} - 1}}. \tag{35}$$

In the limit $k \to 1$ $(L \to \infty)$ the lifetime t_l approaches the ∞ as it should be for a stable field configuration. In the opposite limit of $k \to 0$ lifetime $t_l = \frac{\sqrt{2}}{k}$ is a finite quantity. This is the lifetime of a sphaleron, a configuration at the top of the potential barrier.

ფიზიკა

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შესწავლილია (1+1)-განზომილებიანი სკალარული თეორიის კვანტური თვისებები კომპაქტური სივრცითი ნაწილის მქონე ცილინდრზე. გამოკვლეულია მარტივი მოდელი პოლინომური პოტენციალით, რომელიც შეიცავს ორ განსხვავებულ ვაკუუმს. კლასიკური განტოლებები იძლევა ზუსტი სასრული ენერგიის პერიოდული ველის კონფიგურაციის პოვნის საშუალებას, კერძოდ, ის გამოისახება იაკობის ელიფსური ფუნქციით, რომელიც ითვალისწინებს ამონახსნის პერიოდულობას. ასეთი კონფიგურაციები საინტერესოა კლასიკურ თერმულ და კვანტური გაჟონვის პროცესებს შორის ფაზური გადასვლების კუთხით, რაც მათ

ენერგიაზეა დამოკიდებული. მსგავსი ველის კონფიგურაციების პოვნა ცალსახად რეალურ სივრცე-დროში რთულია. ამდენად, გამარტივებული მოდელები მნიშვნელოვან ინტერესს იძენენ. აგრეთვე, გაანალიზებული მიღებული ველის კონფიგურაციების ფლუქტუაციები დროით გაანალიზებულია გარკვეული გონივრული მიახლოებით. შესაბამისი განტოლების საკუთარი მნიშვნელობები დისკრეტულია და მათ შორის არის უარყოფითი საკუთარი მნიშვნელობა. ამ უარყოფითი საკუთრივ მნიშვნელობის ცხადი გამოსახულება იძლევა მდგომარეობის სიცოცხლის ხანგრძლივობის გამოთვლის საშუალებას. ნაშრომში შესწავლილია შრედინგერის განტოლება კოლექტიური კოორდინატების მეთოდის გამოყენებით, რომლის მეშვეობით შემუშავებულია ტრანსლაციურად ინვარიანტული შეშფოთების თეორია ურთიერთქმედების ზმის უკუხარიხების მიხედვით შეშფოთების თეორიის ნებისმიერ რიგში. კვანტური ფლუქტუაციები კვადრატულ მიახლოებაში გამოთვლილია S-მატრიცა ფეინმანის ტრაექტორიებით ინტეგრალების ჰოლომორფული წარმოდგენის ფარგლებში.

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