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Informatics

Analysis of Gaver's Parallel Systems with Time Redundancy

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Abstract. This paper presents a mathematical model of parallel Gaver systems subject to both partial and complete failures. The job execution is segmented into fixed-length stages, enabling tasks to resume from the failed stage rather than restarting the entire process. The system consists of two identical devices – a primary and a redundant unit – repaired at the same rate. Failure detection is performed either continuously or at the end of each stage. Using the Laplace-Stieltjes transform and integral equations, key system performance metrics such as reliability and average waiting time are derived. The proposed model is particularly relevant for information and computing systems where uninterrupted and reliable operation is critical. *© 2025 Bull. Georg. Natl. Acad. Sci.*

Keywords: redundant systems, time redundancy, partial and complete failures, stage-based job modeling

Introduction

In modern information and computing systems, one of the most critical challenges is ensuring operational reliability of the technical infrastructure. Although significant advances have been made in monitoring and diagnostic tools, it is not always possible to resume information processing precisely from the point of interruption following a failure. This limitation often results in substantial nonproductive time losses, reducing system efficiency and increasing operational costs (Gao et al., 2020; Mazundar, 1979).

One effective solution in such cases is the use of time redundancy – particularly by dividing jobs into independently executable stages. In the event of a failure, only the current stage is repeated, rather than restarting the entire job. This approach minimizes lost effort and allows for more efficient renewal (Khurodze et al., 2017; Khurodze et al., 2022; Khurodze & Kakubava, 2025).

In this study, we analyze a system comprising two identical devices: one primary and one redundant. Both are subject to failures with different rates and are repaired at the same repaired rate. Failures can be detected either continuously or at the end of each stage. Depending on the detection mode, the system resumes the interrupted job from the current stage using the redundant device or the renewed primary unit.

We build a mathematical model that incorporates stage-based job execution, probabilistic failure detection, and redundant replacement. Analytical tools such as Laplace-Stieltjes transforms and generating functions are employed to derive key reliability and performance measures. Furthermore, the model is extended to consider "catastrophic" failures and different control strategies for their detection and recovery.

System Description and Modeling Assumptions

Let us consider the process of completing a job by a service environment consisting of two identical unreliable devices with partially devaluing failures. If a failure is detected in the primary device, replacement is performed by the redundant device from the beginning of the current stage. The system operates in standby mode, i.e. one device is operational, the other is redundant.

Each job consists of one or more stages of constant length τ, and the number of stages for the totality of all requirements is a random variable. The probability that the number of stages in an arbitrary requirement is n, denoted by p_n. The primary device fails with an intensity of α, and the redundant device with an intensity of β.

The redundant device is monitored continuously, while the working device is monitored at the end of each stage. With probability 1, this monitoring detects a failure that may have occurred during the corresponding stage. If a failure is detected, the device undergoes renewal, and the interrupted job is resumed by the redundant device from the beginning of the stage in which the failure occurred-resulting in the loss of completed execution for that stage. The replacement by redundant device is instantaneous. The repair rate for both devices is equal to μ.

We consider the functions:

$H(x)$ – the distribution function of the job execution time by a single device, assuming no failures occur during service.

$R_{ij}(x) = P\{$At the end of the time interval $[0, x]$, the system is in state j if it was in state i at the beginning of this interval. Accordingly, for $i = 1$, the redundant device is assumed to have failed; for $i = 2$, it is considered operational.$\}$ (Sandler, 2012).

$H_{ij}(x) = P\{$The job execution time is less than x; the Gaver's parallel system is in state j at the end of the job, given that it was in state i at the beginning of the execution$\}$.

$H_{ij}(n,x) = P$ $\{$The execution time of a requirement consisting of n stages is less than x; at the end of execution, the servicing device complex is in state j, given that it was in state i at the beginning of execution$\}$.

The function $H_{ij}(n,x)$ is defined as the probability of the event $A_{ij}(n,x)$

$$H_{ij}(n,x) = P\{A_{ij}(n,x)\}.$$

We obtain the Laplace transform expression for the function $H_{ij}(x)$ under the above assumptions.

Theorem 1. The functions $H_{ij}(1,t)$ satisfy a system of integral equations:

$$H_{ij}(1,t) = \int_0^t e^{-\alpha x} R_{ij}(x) dH(x) + \int_0^t (1 - e^{-\alpha x}) R_{i2}(x) H_{1j}(1, t-x) dH(x) +$$
$$+ \int_0^t (1 - e^{-\alpha x}) R_{i1}(x) dH(x) \int_0^{t-x} \mu e^{-\mu v} H_{1j}(1, t-x-v) dv, \quad i,j = 1,2. \qquad (1)$$

Proof: Let us obtain the equation for $H_{11}(1,t)$. Event $A_{11}(1,t)$ can be decomposed into the following disjoint events:

1. At the moment x $(0 \leq x \leq t)$, the stage execution ends, no failure occurred during this time, and at this moment the redundant device is defective. The probability of this complex event integrated over the entire range of the variable x, is $\int_0^t e^{-\alpha x} R_{11}(x) dH(x)$.

2. At the moment x $(0 \leq x \leq t)$, the execution of the stage ends. At the end of the stage, it is found that the operating device has failed, at this moment the redundant device is in good condition and it continues execution, after which the event $A_{11}(1, t-x)$ occurs.

The probability of a similar event, summed over all possible values of x, is $\int_0^t (1 - e^{-\alpha x}) R_{12}(x) dH(x)\, H_{11}(1, t-x) dH(x)$.

3. At the moment x $(0 \leq x \leq t)$, the execution of the stage ends, at the end of the stage it is discovered that a failure of the working device has occurred; at this moment, the redundant device is repaired, it will be renewed in time v $(0 \leq v \leq t - v)$ and continues the execution of the stage again, then the event $A_{11}(t - x - v)$.

The probability of this event, considering all possible values of the variables x and v, is $\int_0^t (1 - e^{-\alpha x}) R_{11}(x) dH(x) \int_0^{t-x} \mu e^{-\mu v} H_{11}(1, t - x - v) dv$.

As a result, we get

$$H_{11}(1,t) = \int_0^t e^{-\alpha x} R_{11}(x) dH(x) + \int_0^t (1 - e^{-\alpha x}) R_{12}(x) H_{11}(1, t-x)\, dH(x) +$$
$$+ \int_0^t (1 - e^{-\alpha x}) R_{11}(x) dH(x) \int_0^{t-x} \mu e^{-\mu v} H_{11}(1, t - x - v) dv,$$

which coincidence the equation in system (1) corresponding to $j = 1$.

The equations for other values of i and j are obtained similarly.

Applying the Leplace-Stiltjes transformations to the obtained equations, we get the corresponding system of algebraic equations with respect to $\bar{h}_{i,j}(1,s)$ (since the Laplace transformation of the functions $h_{i,j}(n,t)$ coincide with the Laplace-Stiltjes transformation of the functions $H_{i,j}(n,t) = \int_0^t h_{ij}(n,x) dx$).

$$\bar{h}_{11}(1,s) = \frac{a_1(s,\beta,\mu)}{1 - b_1(s)};$$

$$\bar{h}_{12}(1,s) = \frac{a_2(s,\beta,\mu)}{1 - b_1(s)};$$

$$\bar{h}_{21}(1,s) = a_2(s,\beta,\mu) + \frac{a_1(s,\beta,\mu)}{1 - b_1(s)};$$

$$\bar{h}_{22}(1,s) = a_1(s,\beta,\mu) + \frac{a_2(s,\beta,\mu)}{1 - b_1(s)}.$$

The following notations are applied here:

$$a_1(s,\beta,\mu) = \frac{\left[\beta \bar{h}(s+\alpha) + \mu \bar{h}(s + \alpha + \beta + \mu)\right]}{(\beta + \mu)};$$

$$a_2(s,\beta,\mu) = \frac{\mu}{\beta + \mu}\left[\bar{h}(s+\alpha) - \bar{h}(s + \alpha + \beta + \mu)\right];$$

$$b_1(s) = \frac{\mu}{(\beta + \mu)(s + \mu)}\left\{(s + \beta + \mu)\left[\bar{h}(s) - \bar{h}(s+\alpha)\right] - \right.$$
$$\left. -s\left[\bar{h}(s + \beta + \mu) - \bar{h}(s + \alpha + \beta + \mu)\right]\right\};$$

$$b_2(s)=\frac{1}{(\beta+\mu)(s+\mu)}\{\mu(s+\beta+\mu)[\bar{h}(s)-\bar{h}(s+\alpha)]+$$
$$+\beta s[\bar{h}(s+\beta+\mu)-\bar{h}(s+\alpha+\beta+\mu)]\},$$

In the equations above $\bar{h}(s)=e^{-s\tau}$, since $H(x)=\begin{cases}1, & if\ \ x<\tau\\ 0 & if\ \ x<\tau.\end{cases}$

Theorem 2. For any n, $(n=2,3,\dots)$, the function $H_{i,j}(n,t)$, satisfies the following system of integral equations

$$H_{i,j}(n,t)=\int_0^t H_{1j}(n-1,t-x)dH_{i1}(1,u)+\int_0^t H_{2j}(n-1,t-x)dH_{i2}(1,x), i,j=1,2. \quad (2)$$

The proof of this theorem follows the same approach as that of the Theorem 1.

After the Laplace-Stieltjes transformation, from (2) we obtain a system of algebraic equations

$$\bar{h}_{ij}(n,s)=\bar{h}_{i1}(1,s)\bar{h}_{1j}(n-1,s)\bar{h}_{i2}(1,s)\bar{h}_{2j}(n-1,s), i,j=1,2. \quad (3)$$

It holds for $n-1$ if we introduce the notation $\bar{h}_{1j}(0,s)=\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol.

Although the solution to system (2) is straightforward, it is more insightful to derive the result directly from System (3) using the supplementary variable method (Cox, 1955).

Let us introduce the concept of "catastrophe". Suppose that disasters with intensity $s>0$ occur during the execution of a requirement. Then $\bar{h}_{1j}(n,s)=\int_0^\infty e^{-st}dh_{ij}\,(n,t)$ expresses the conditional probability of an event, which is that at the end of the execution of the requirement consisting of n stages, the complex of servicing devices is in state j and no "catastrophe" occurred during execution, provided that at the beginning of execution the complex of devices is in state i. In other words, $\bar{h}_{ij}(n,s)=P\{A_{i,j}(n,\infty)\cap B_n\}$, where B_n is an event consisting in the fact that no disasters occur during the execution of the n stage requirement.

The following relation can be readily obtained:

$$A_{ij}(n,\infty)\cap B_n=\{A_{i1}(1,\infty)\cap B_1\}\cap\{A_{1j}(n-1,\infty)\cap B_{n-1}\}+$$
$$+\{A_{i2}(1,\infty)\cap B_1\}\cap\{A_{2j}(n-1,\infty)\cap B_{n-1}\}. \quad (4)$$

In short, equation (4) states the following: for a 'catastrophe' not to occur during n stages, it is necessary and sufficient that it does not happen at the first stage and also does not occur during the subsequent $n-1$ stages.

Let us introduce generating functions $h_{ij}(z,s)=\sum_{n=1}^{\infty} z^n\bar{h}_{ij}\,(n,s)$ $\quad |z|\le 1,\ i,j=1,2.$

Multiplying both sides of equation (3) by z^n and summing over n, we obtain the following system of algebraic equations with respect to $h_{ij}(z,s), i,j=1,2.$

$$h_{ij}(z,s)=z[\bar{h}_{i1}(1,s)\delta_{1j}+\bar{h}_{i2}(1,s)\delta_{2j}]+z\bar{h}_{i1}(1,s)h_{1j}(z,s)+z\bar{h}_{i2}(1,s)h_{2j}(z,s).$$

The solution of this system has the form:

$$h_{ii}(z,s)=\frac{z\bar{h}_{ii}(1,s)[1-z\bar{h}_{jj}(1,s)]+z^2\bar{h}_{ij}(1,s)\bar{h}_{ji}(1,s)}{D(z,s)}, \qquad i\ne j, i,j=1,2.$$

$$h_{ij}(z,s)=\frac{z\bar{h}_{ij}(1,s)}{D(z,s)};$$

Here

$$D(z,s) = \left[1 - z\bar{h}_{11}(1,s)\right]\left[1 - z\bar{h}_{22}(1,s)\right] - z^2\bar{h}_{12}(1,s)\bar{h}_{21}(1,s).$$

Suppose, the number of application stages has a geometric distribution with the parameter p. In this case $p_n = p(1-p)^{n-1}$, $0 < p < 1$.

According to the formula of total probability $H_{ij}(t) = \sum_{n=1}^{\infty} p_n H_{ij}(n,t)$, or in an operational form $\bar{h}_{ij}(s) = \sum_{n=1}^{\infty} p_n \bar{h}_{ij}(n,s)$, take into account that $H_{ij}(t) = \int_0^t h_{ij}(t)dx$.

The last equality in our case takes the form:

$$\bar{h}_{ij}(s) = \sum_{n=1}^{\infty} p(1-p)^{n-1}\bar{h}_{ij}(n,s) = \frac{p}{1-p}\sum_{n=1}^{\infty}(1-p)^n\bar{h}_{ij}(n,s) = \frac{p}{1-p}h_{ij}(1-p,s) \quad (5)$$

It is apparent that the functions $\bar{h}_{ij}(s)$ are obtained from the function $\bar{h}_{ij}(z,s)$ by replacing z with $1-p$ and multiplying the result by $\frac{p}{1-p}$.

Model Extension and Implications for Advanced Failure Control

Further generalization of the considered model can proceed in several directions. One particularly relevant extension is to account for the impact of self-canceling failures on the operation of a redundant system. We will explore control strategies for detecting such failures, assuming they are identified through continuous monitoring, after which the current stage is serviced again.

We denote by $F(t)$ the distribution function of the service time of one stage with length τ, provided that stable failures do not occur in the servicing device. Let us define the expressions $\bar{F}(s)$ of the above option. We keep all the designations and in addition, we assume that the failure rate for the primary device is denoted by γ.

Theorem 3. The function $F(t)$ satisfies the following integral equation,

$$F(t) = \int_0^t e^{-\gamma x}\, dH(x) + \int_0^t \gamma e^{-\gamma x}[1 - H(x)]\, F(t-x)dx. \quad (6)$$

Proof: For $t \le \tau, F(t) = 0$, which satisfies (6), since $\int_0^t e^{-\gamma x} dH(x) = 0$ at $t \le \tau$. This follows from the definition $H(x)$.

When $t > \tau$, a complex event $A(t) = \{$the execution time of one stage of length τ is less than t without sustained failures of the device$\}$, can be represented as the sum of the following two events:

1. $A_1(t) = \{$no failures occur during execution$\}$, with probability $P\{A_1(t)\} = e^{-\gamma\tau} = \int_0^t e^{-\gamma x} dH(x)$;

2. $A_2(t) = \{$at the moment $x, (0 < x < \tau)$, a failure occurs, after that an event occurs $A(t-x)\}$. Considering all possible values for the variable x, it turns out

$$P\{A_2(t)\} = \int_0^t \gamma e^{-\gamma x} F(t-x)dx = \int_0^t \gamma e^{-\gamma x}[1 - H(x)]F(t-x)dx.$$

Thus, it has been shown that $F(t)$ satisfies equation (6) for all t.

Now, it is easy to obtain the Laplace-Stieltjes transform $\bar{F}(s)$ for $F(t)$

$$\bar{F}(s) = \frac{s+\gamma}{se^{(s+\gamma)\tau}\gamma}\ . \quad (7)$$

There may be situations in which failures are detected either at the end of each stage - after which the stage is reprocessed - or through a repeated counting method, where failures are confirmed once two consecutive execution results for the same stage coincide.

In these cases, the Laplace-Stieltjes transform for $F(t)$ takes the following form

$$\bar{F}(s) = \frac{1}{e^{(s+\gamma)\tau} - e^{\gamma\tau} + 1}. \tag{8}$$

and

$$\bar{F}(s) = \frac{1}{\left[e^{(s+\gamma)\tau} - e^{\gamma\tau} + 1\right]^2}. \tag{9}$$

After the expressions for $\bar{F}(s)$ are obtained, the transition to the general case is not difficult. In fact, it is enough to put $F(x)$ in the system of equations (1) instead of the function $H(x)$. All other transformations are transferred purely mechanically. Thus, a model of a redundant system with heterogeneous failures and various methods of monitoring, detecting and correcting errors in information processing and transmission systems is obtained.

Conclusion

This paper presents a mathematical model for analyzing redundant systems subject to both partial and complete failures. By dividing job execution into stages and using probabilistic methods, the model helps calculate important performance measures like waiting time and reliability. It also considers different failure detection strategies and includes the possibility of sudden "catastrophic" failures. The proposed model can be applied in information and computing systems where reliable and continuous operation is critical, such as in data centers, communication networks, etc.

ინფორმატიკა

გავერის პარალელური სისტემების ანალიზი დროითი რეზერვის გათვალისწინებით

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§ ბათუმის შოთა რუსთაველის სახელმწიფო უნივერსიტეტი, ზუსტ მეცნიერებათა და განათლების ფაკულტეტი, საქართველო

ნაშრომში განხილულია გავერის პარალელური სისტემების მათემატიკური მოდელი, რომელიც განიცდის როგორც ნაწილობრივ, ისე სრულ მტყუნებას. სამუშაო პროცესი დაყოფილია გარკვეული სიგრძის ეტაპებად, რაც საშუალებას იძლევა მტყუნებების შემთხვევაში დავალების შესრულება გაგრძელდეს მიმდინარე ეტაპიდან და არა თავიდან. სისტემა შედგება ორი იდენტური მოწყობილობისაგან – ძირითადი და სარეზერვო, რომელთა მტყუნება ხდება სხვადასხვა ინტენსივობით, აღდგენა კი – ერთნაირი ინტენსივობით. მტყუნებათა გამოვლენა ხდება ან უწყვეტად, ან ყოველი ეტაპის ბოლოს. ლაპლას-სტილტიესის გარდაქმნისა და ინტეგრალური განტოლებების გამოყენებით მიღებულია სისტემის ფუნქციონირების ძირითადი მახასიათებლები, როგორიცაა სისტემის საიმედოობა და ლოდინის საშუალო დრო. განხილული მოდელი აქტუალურია ინფორმაციულ და გამოთვლით სისტემებში, სადაც უწყვეტი და საიმედო მუშაობა სასიცოცხლოდ მნიშვნელოვანია.

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