საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 19(193), №4, 2025
BULLETIN OF THE NATIONAL ACADEMY OF SCIENCES OF GEORGIA, vol. 19(193), no. 4, 2025

Mathematics

Approximate Solution to One Fourth-Order Nonlinear Parabolic-Type Integro-Differential Equation

Temur Jangveladze[*], Besiki Tabatadze[**], Teimuraz Chkhikvadze[§], Zurab Kiguradze[§§]

[*] *I. Vekua Institute of Applied Mathematics, Georgian Technical University, Tbilisi, Georgia*
[**] *European University; Georgian American University, Tbilisi, Georgia*
[§] *Ivane Javakhishvili Tbilisi State University, Georgia*
[§§] *Missouri University of Science and Technology, USA*

(Presented by Academy Member Elizbar Nadaraya)

Abstract. Many processes are described by nonlinear integro-differential models. The initial-boundary value problem for the fourth-order nonlinear parabolic integro-differential equation is considered. It is well-known that the essential difficulties arise in the processes of constructing, investigating and implementing the numerical algorithms for the considered model. The goal of this paper is to study the problem under consideration using machine learning methods. *© 2025 Bull. Natl. Acad. Sci. Georg.*

Keywords: nonlinear integro-differential equations, numerical solution, machine learning algorithms, deep neural networks

Many processes are described by nonlinear integro-differential models (Friedman, 1964; Ladyzhenskaya et al., 1968; Lions, 1969; Lakshmikantham et al., 1995). In (Gordeziani et al., 1983), the reduction of the well-known Maxwell system (Landau et al., 1960) of differential equations to the form of integro-differential equations was performed. The mentioned reduced model has the following form:

$$\frac{\partial H}{\partial t} = -rot\left[\left(a\int_0^t |rotH|^2\, d\tau\right) rotH\right], \tag{1}$$

where H is the vector of the magnetic field. Some qualitative and structural properties of solutions of (1) type systems are established in many works (see, e.g., Gordeziani et al., 1983; Jangveladze, 1983; Jangveladze, 1985; Laptev, 1988; Lin et al., 1992; Jangveladze, 1997; Jangveladze, 1998; Jangveladze et al., 2002; Dzhangveladze et al., 2007; Dzhangveladze et al., 2008; Jangveladze et al., 2008; Aptsiauri et al., 2012; Jangveladze et al., 2013; Hecht, et al., 2018; Chkhikvadze, 2021; Jangveladze et al., 2024). For more detail information see (Jangveladze et al., 2015; Jangveladze, 2019) and references therein.

The present work discusses a natural mathematical generalization of the scalar analog of the integro-differential model (1). In particular, the corresponding fourth-order integro-differential equation is investigated and some properties of the posed initial-boundary value problems are studied.

The computer implementations were carried out using both iterative methods (Jangveladze et al., 2015; Jangveladze, 2019) and deep neural networks (Blechschmidt et al., 2021; Raissi et al., 2017; Raissi et al., 2017; Raissi et al., 2019). Numerical experiments were conducted, and the obtained results were compared.

Nonlinear integro-differential equations arise in various physical and engineering contexts, including the theory of heat conduction in materials with memory, viscoelasticity, electrodynamics, and other fields. In particular, the reduction of Maxwell equations to a form of integro-differential models has motivated a deeper investigation of their scalar and vector analogues. These models often exhibit complex behaviors due to their nonlinear and nonlocal nature.

In this study, we consider a fourth-order parabolic-type integro-differential equation as a scalar analogue of the system derived from Maxwell equations. Such models are relevant for describing long-term evolution in physical systems where both diffusion and memory effects play significant roles.

The present work aims to formulate the problem rigorously, discuss the analytical challenges, and provide computational strategies based exclusively on deep learning methods, particularly, fully connected feedforward neural networks, for its approximate solution.

We consider the following nonlinear integro-differential equation of parabolic type:

$$\frac{\partial u}{\partial t}+\frac{\partial^2}{\partial x^2}\left\{\left[1+\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)^2 d\tau\right]\frac{\partial^2 u}{\partial x^2}\right\}=f(x,t), \tag{2}$$

where $u(x,t)$ is a function defined on the domain $[0,1]\times[0,T]$, and T is a positive constant. The equation characterizes processes with spatial diffusion and hereditary (memory) effects.

The boundary and initial conditions are:

$$u(0,t)=u(1,t)=0, \tag{3}$$

$$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(1,t)=0, \tag{4}$$

$$u(x,0)=u_0(x). \tag{5}$$

In equation (2) and in the initial condition (5), f and u_0 are given functions of their arguments.

Our objective is to approximate the solution using a deep learning-based approach. Specifically, we employ a fully connected feedforward neural network (FNN) as a function approximator, trained to minimize the residuals of the integro-differential equation and satisfy the initial and boundary conditions.

It is possible to prove that the solution of the problem (2)-(5) is stable with respect to the right-hand side f and the initial condition u_0.

$$\|u\|^2 \le \int_0^t \|f(t)\|^2 d\tau + \|u_0\|^2,$$

where $\|\cdot\|$ denotes the norm of the space $L_2(0,\ 1)$. It is also possible to prove the uniqueness of the solution of the problem (2)-(5) (see, e.g. Chkhikvadze, 2021).

Below we describe the structure of the neural network, the loss function formulation, and the training methodology applied to solve the problem (2)-(5).

Deep learning is a subset of machine learning that uses artificial neural networks with multiple layers to model complex patterns in data. Unlike traditional machine learning algorithms, deep learning methods can automatically learn representations from data through hierarchical feature extraction.

One of the most fundamental architectures in deep learning is the fully connected FNN. An FNN consists of an input layer, one or more hidden layers, and an output layer. Each neuron in a given layer is connected to every neuron in the subsequent layer. The nonlinear activation functions are applied to the weighted sums of neuron inputs, enabling the network to approximate highly complex functions.

In this study, we use the FNN as a function approximator for the solution of the integro-differential equation. The network is trained to minimize a loss function that includes the residual of the governing equation, as well as the discrepancy from the initial and boundary conditions. This framework allows the solution to satisfy both the equation and its associated constraints in a data-driven manner.

This approach is particularly suitable for the integro-differential equation considered in this paper, which contains nonlocal and nonlinear terms that are difficult to handle using classical numerical schemes. The flexibility and approximation power of FNN allow us to encode both the equation and its initial-boundary conditions into a unified loss function. This enables the neural network to learn the solution over the entire spatio-temporal domain without requiring mesh generation or explicit discretization of the integral term.

To train the neural network for approximating the solution of the integro-differential equation, we define a total loss function that guides the learning process. This loss function combines multiple components: one that measures how well the neural network satisfies the integro-differential equation, and others that ensure the network respects the initial and boundary conditions.

During training, sample points are selected throughout the spatial and temporal domain. The loss is computed by evaluating how much the neural network's output deviates from the governing equation and prescribed conditions at those points. The network parameters are updated iteratively to minimize this total loss.

This approach allows the neural network to learn a solution that is consistent across the entire domain without relying on traditional grid-based discretization. The training is performed using optimization algorithms such as stochastic gradient descent or the Adam optimizer, often relying on automatic differentiation to compute necessary gradients.

To evaluate the effectiveness of the proposed deep learning approach, we conducted numerical experiments on the model equation. The spatial domain and time interval were discretized by uniformly sampling collocation points. The neural network was trained using the Adam optimizer, with the learning rate adjusted through scheduling.

The loss function convergence during training showed a steady decrease, indicating successful learning. After training, the network output was compared to known benchmark solutions or numerical approximations obtained using traditional methods, where available. The comparison demonstrated that the deep learning model accurately captured both the transient and steady-state behaviors of the solution.

Additionally, the learned solution maintained stability and smoothness across the domain. In particular, the nonlocal memory effect embedded in the integral operator was effectively handled by the neural architecture without explicit discretization of the integral term.

These results support the feasibility of using neural networks to approximate complex nonlinear integro-differential problems, offering an alternative to classical mesh-based methods.

In our experiment in problem (2)-(5), we have chosen the exact solution with the corresponding right-hand side function $f(x,t)$. In this experiment the exact solution has the following form:

$$U(x,t) = x^2 (1-x)^2 e^{-x-t},$$

with appropriate initial condition.

The neural network model used for the numerical simulation was implemented using the TensorFlow framework. To build and train such a model, one typically needs a Python programming environment with libraries such as TensorFlow or PyTorch, along with tools for scientific computing (NumPy, SciPy) and visualization (Matplotlib). Training can be done on a standard personal computer; however, for larger problems or faster training, a machine with a GPU (graphics processing unit) is recommended.

A suitable platform for developing these models includes Jupyter Notebook at Google Colab, which offers interactive environments with access to computational resources. Google Colab is particularly useful as it provides free access to GPUs and supports all necessary Python libraries for deep learning development.

To further illustrate the accuracy of the model, Table 1 presents numerical values comparing the exact and predicted solutions for selected values of x and t.

Table 1. Comparison between exact and predicted solutions

x	*t*	Exact *U(x,t)*	Predicted *U(x,t)*	Absolute Error
0.2	0.7	0.012713	0.013150	0.000438
0.4	0.7	0.028603	0.029312	0.000708
0.6	0.7	0.028603	0.029312	0.000708
0.8	0.7	0.012713	0.013150	0.000438

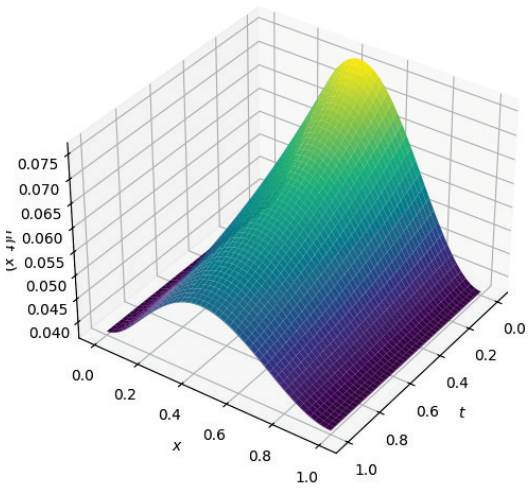

Fig. 1. Approximate solutions.

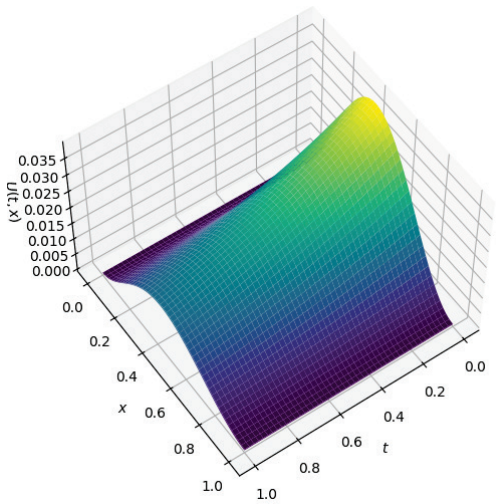

Fig. 2. Exact solutions.

To visually assess the quality of the neural network's prediction, Fig. 1 displays the approximate solutions and Fig. 2 shows exact solutions as 3D surface plots. These plots clearly demonstrate that the neural network's approximation closely follows the true solution profile across the spatio-temporal domain. Minor deviations are visible but remain within acceptable bounds, confirming the model's robustness and reliability.

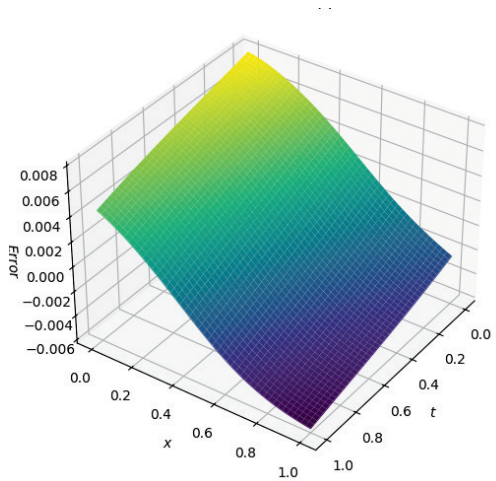

Fig. 3. Difference between exact and approximate solutions.

For further validation of the accuracy of the neural network model, Fig. 3 illustrates the distribution of the absolute error between the exact and predicted solutions over the spatio-temporal domain. The error remains small and smooth across the entire domain, indicating the model's high level of precision and its ability to generalize the solution structure effectively.

This study presents a deep learning-based approach for solving a class of fourth-order nonlinear parabolic-type integro-differential equations. The proposed method utilizes a fully connected feedforward neural network to approximate the solution, incorporating the governing equation, initial, and boundary conditions into a unified loss function.

The numerical experiments demonstrate that the neural network model accurately captures the behavior of the exact solution, with low approximation error and good generalization across the domain. Both visualization and quantitative analysis confirm the reliability and effectiveness of the approach.

Overall, the results support the potential of deep learning methods as a viable and powerful alternative to classical mesh-based numerical schemes, especially for solving complex integro-differential problems involving nonlocal and nonlinear phenomena.

Acknowledgements

This work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG) [grant number FR-21-2101].

მათემატიკა

მეოთხე რიგის ერთი არაწრფივი პარაბოლური ტიპის ინტეგრალურ-დიფერენციალური განტოლების მიახლოებითი ამოხსნა

თ. ჯანგველაძე*, ზ. ტაბატაძე, თ. ჩხიკვაძე§, ზ. კიღურაძე§§**

** ივანე ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი, ი. ვეკუას სახ. გამოყენებითი მათემატიკის ინსტიტუტი; საქართველოს ტექნიკური უნივერსიტეტი, მათემატიკის დეპარტამენტი, საქართველო*
*** ევროპის უნივერსიტეტი; საქართველოს ამერიკული უნივერსიტეტი, თბილისი, საქართველო*
§ ივანე ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი, საქართველო
§§ მისურის მეცნიერებისა და ტექნოლოგიის უნივერსიტეტი, აშშ

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

მრავალი პროცესი აღიწერება არაწრფივი ინტეგრალურ-დიფერენციალური მოდელებით. მეოთხე რიგის ერთი არაწრფივი პარაბოლური ინტეგრალურ-დიფერენციალური განტოლებისათვის განხილულია საწყის-სასაზღვრო ამოცანა. კარგადაა ცნობილი რომ არსებითი სირთულეები წარმოიქმნება აღნიშნული მოდელისთვის რიცხვითი ალგორითმების აგების, გამოკვლევისა და რეალიზაციის პროცესში. ამ ნაშრომის მიზანია შევისწავლოთ განსახილველი ამოცანა მანქანური სწავლის მეთოდის გამოყენებით.

REFERENCES

Aptsiauri, M. M., Jangveladze, T. A. & Kiguradze, Z. V. (2012). Asymptotic behavior of the solution of a system of nonlinear integro-differential equations. *Differ. Uravn., 48*(1), 70-78; translation in *Differ. Equ., 48*(1), 72-80.

Blechschmidt, J., Ernst, O.G. (2021). Three ways to solve partial differential equations with neural networks – a review. *GAMM-Mitteilungen*., *44*(2), e2021000062021.

Chkhikvadze, T. (2021). On one nonlinear integro-differential parabolic equation. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.*, (35), 19-22.

Friedman, A. (1964). *Partial differential equations of parabolic type.* Prentice-Hall.

Gordeziani, D., Jangveladze, T. & Korshiya, T. (1983). Existence and uniqueness of the solution of a class of nonlinear parabolic problems. *Differ. Uravn., 19*(7), 1197-1207. *Differ. Equ., 19*(7), 887-895.

Hecht, F., Jangveladze, T., Kiguradze, Z. & Pironneau, O. (2018). Finite difference scheme for one system of nonlinear partial integro-differential equations. *Appl. Math. Comput.* (328), 287-300.

Jangveladze, T. (1997). On one class of nonlinear integro-differential parabolic equations. *Semin. I. Vekua Inst. Appl. Math. Rep.* (23), 51-87.

Jangveladze, T. (1998). Convergence of a difference scheme for a nonlinear integro-differential equation. *Proc. I. Vekua Inst. Appl. Math.,* (48), 38-43.

Jangveladze, T. (2019). Investigation and numerical solution of nonlinear partial differential and integro-differential models based on system of Maxwell equations. *Mem. Differential Equations Math. Phys.* (76), 1-118.

Jangveladze, T. A. & Kiguradze, Z. V. (2007). On the stabilization of solutions of an initial-boundary value problem for a nonlinear integro-differential equation. *Differ. Uravn., 43*(6), 833-840; translation in *Differ. Equ., 43*(6), 854-861.

Jangveladze, T. A. & Kiguradze, Z. V. (2008). Asymptotic behavior of the solution of a nonlinear integrodifferential diffusion equation. *Differ. Uravn., 44*(4), 517-529; translation in *Differ. Equ., 44*(4), 538-550.

Jangveladze, T. and Kiguradze, Z. (2002). Estimates of a stabilization rate as $t \to \infty$ of solutions of a nonlinear integro-differential equation. *Georgian Math. J.,* (9), 57-70.

Jangveladze, T. and Kiguradze, Z. (2008). Large time behavior of solutions to one nonlinear integro-differential equation. *Georgian Math. J.,* (15), 531-539.

Jangveladze, T., Kiguradze, Z. & Chkhikvadze, T. (2024). Application of Deep Neural Network for numerical approximation for averaged nonlinear integro-differential equation. *Bulletin of TICMI., 28*(1), 3-10.

Jangveladze, T., Kiguradze, Z. & Neta, B. (2015). Numerical solution of three classes of nonlinear parabolic integro-differential equations. Elsevier/Academic Press, Amsterdam.

Jangveladze, T., Kiguradze, Z., Neta, B. & Reich, S. (2013). Finite element approximations of a nonlinear diffusion model with memory. *Numer. Algor*. (64), 127-155.

Jangveladze, T.A. (1983). First boundary-value problem for a nonlinear equation of parabolic type. *Dokl. Akad. Nauk SSSR, 269*(4), 839-842. English translation: *Soviet Phys. Dokl.,* 1983, *28*(4), 323-324.

Jangveladze, T.A. (1985). A nonlinear integro-differential equation of parabolic type. *Differ. Uravn., 21*(1), 41-46; translation in *Differ. Equ., 21*(1), 32-36.

Ladyzhenskaya, O. A., Solonnikov, V. A. & Uralceva N. N. (1968). Linear and quasilinear equations of parabolic type. Moscow.

Lakshmikantham, V. & Rao, M. R. M. (1995). Theory of integro-differential equations, CRC Press.

Landau, L., & Lifschitz, E. (1960). *Electrodynamics of Continuous Media.* 8(1). Pergamon Press.

Laptev, G. I. (1988). Quasilinear parabolic equations that have a Volterra operator in the coefficients. *Mat. Sb. (N.S.), 136*(178), 530-545; translation in *Math. USSR-Sb., 64*(2), 527-542.

Lin, Y. P. & Yin, H.-M. (1992). Nonlinear parabolic equations with nonlinear functionals. *J. Math. Anal. Appl., 168*(1), 28-41.

Lions, J.-L. (1969). *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*. (in French) Dunod; Gauthier-Villars, Paris.

Raissi, M., Perdikaris, P. & Karniadakis, G.E. (2017). Physics informed deep learning (part I): Datadriven solutions of nonlinear partial differential equations. *arXiv 1711.10561*.

Raissi, M., Perdikaris, P. & Karniadakis, G.E. (2017). Physics informed deep learning (part II): Datadriven discovery of nonlinear partial differential equations. *arXiv 1711.10566*.

Raissi, M., Perdikaris, P. & Karniadakis, G.E. (2019). Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, (378), 686-707.

Received July, 2025