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Mathematics

The Linear Regression Model’s Coefficients Consistent Estimators of Parameters and Consistent Hypothesis Testing Criteria for Gaussian Statistical Structure

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Abstract. Statistical method should be used to determine probabilistic characteristics. Among the problems of statistics there is a class of problems, in which the number of observation is unique. Despite the uniqueness of observations in many cases it is possible to reliably determine the values of unknown distribution parameters or to reliably choose one of infinite numbers of competing hypotheses about the exact form of the distribution. In the case when parameter is reliably determined by one observation, it is said that for it there exists a consistent estimate of parameter. In the paper, we define Gaussian statistical structure necessary and sufficient conditions for the existence of consistent estimators of parameters and consistent hypothesis testing criteria for Gaussian statistical structure. *© 2026 Bull. Natl. Acad. Sci. Georg.*

Keywords: consistent estimator, consistent criteria, strongly separable, orthogonar, weakly separable, statistical structure

Introduction

Most econometric models are described by regression equations. In general, this type of model has a form $y = f(x_1, x_2, ..., x_n) + u$, where y is dependent variable, $x_1, x_2, ..., x_n$ are independent variables, and u is a random variable. When the number of independent variables is $n \geq 2$, then the equation given here is called a multiple regression model. However, if the econometric model contains only one independent variable, i.e. if $y = f(x) + u$ then we have a pairwise model.

We consider statistical relationship. In particular, we consider the linear regression model $y = \beta_0 + \beta_1 x + u,$ y is represented by a non-random component $\beta_0 + \beta_1 x$ and a random component u. In addition, u is a normally distributed with a zero expectation and an unknown variance σ_u^2. In this paper, Gaussian statistical structures to estimate the unknown parameters, $\beta_0, \beta_1, \sigma_u^2$ are considered. For strongly separable Gaussian statistical structures, consistent estimates and consistent hypothesis testing criteria for σ_u^2 are constructed. Then consistent estimates and consistent hypothesis testing criteria for the parameters β_0 and β_1 are constructed.

Let (E,S) be a measurable space with a given family of probability measures $\{\mu_i, i \in I\}$. Some definition from (Ibramkhalilov and Skorokhod, 1980; Jech, 2003; Zerakidze, 2005; Zerakidze, 2008; Zerakidze and Patsatsia, 2018).

Definition 1. An object $\{E, S, \mu_i, i \in I\}$ is called a statistical structure.

Definition 2. An statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular), if $\{\mu_i, i \in I\}$ consists of pairwise singular measures (i.e. $\mu_i \perp \mu_j, \forall i \neq j$).

Definition 3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called a weakly separable, if there exists a family of S - measurable set $\{X_i, i \in I\}$ such that the relations are fulfilled:

$$\mu_i(X_{i'}) = \begin{cases} 1, & if \quad i = i'; \\ 0, & if \quad i \neq i'. \end{cases} \qquad (i, i' \in I).$$

Definition 4. A statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is called strongly separable if there exist the family S - measurable sets $\{X_i, i \in I\}$, such that:

1) $\mu_i(X_{i'}) = 1, \forall i \in I$;

2) $X_i \bigcap X_j = \varnothing, \ \forall i \neq j$.

Let I be the set of parameters and let $B(I)$ be σ -algebra of subsets of I, which contains all finite subsets of I.

Definition 5. We will say that the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ admits a consistent estimate of parameters $i \in I$, if there exists at least one measurable mapping

$$f : (E, S_1) \to (I, B(I)),$$

such that

$$\mu_i(\{x : f(x) = i\}) = 1, \ \forall i \in I.$$

Let H be a set of hypotheses and $B(H)$ be σ -algebra of subsets of H which contains all finite subsets of H.

Definition 6. We will say that the statistical structure $\{E, S, \mu_h, h \in H\}$ admits a consistent criterion (same "Generalization Neiman-Pearson", z - criterion) for hypotheses testing, if there exists at least one measurable mapping

$$\delta : (E, S) \to (H, BH)),$$

such that

$$\mu_h(\{x : \delta(x) = h\}) = 1, \ \forall h \in H.$$

Definition 7. The probability $\alpha_h(\delta) = \mu_h(\{x : \delta(x) \neq h\})$ is called the probability of error of *h*-th kind for the given criterion δ.

The density of Gaussian (normal) low is determined by the equality

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma_u^2}} .$$

Let μ be the probability measures given $\{R, B(R)\}$ by the formula $\mu(A) = \int_A f(x)\mu(dx)$.

Definition 8. An object $\{E, S, \mu_i, i \in I\}$ is called Gaussian statistical structure.

The linear Regression Model for a countable statistical structure.

Consider the linear regression model

$$y = \beta_0 + \beta_1 x + u, \text{ with } u \sim N(0, \sigma_u^2), \tag{1}$$

where σ_u^2, β_0, β_1 are unknown parameters.

Let $\{\mu_{\sigma_u^2}, \sigma_u^2 \in Q_+\}$ (where Q_+ - the set of nonnegative rational numbers) be Gaussian probability measures defined on the measurable space $\left\{(-\infty, +\infty), B(-\infty, +\infty), \mu_{\sigma_u^2}, \sigma_u^2 \in Q_+\right\}$.

Theorem 1. In order for Gaussian statistical structure $\left\{(-\infty, +\infty), B(-\infty, +\infty), \mu_{\sigma_u^2}, \sigma_u^2 \in Q_+\right\}$ to admit a consistent estimator of parameters $\sigma_u^2 \in Q_+$ in the theory (ZFC) it is necessary and sufficient that this statistical structure be strongly separable (Definition 4).

Proof. Necessity. The existence of a consistent estimators of parameters, means that there exist at least one measurable mapping $f : \{(-\infty, +\infty), B(-\infty, +\infty)\} \to \{[0+\infty), B[0, +\infty)\}$, such that

$\mu_{\sigma_u^2}(\{x : f(x) = \sigma_u^2\}) = 1, \ \forall \sigma_u^2 \in Q_+$.

Denoting $Z_{\sigma_u^2} = \{x : f(x) = \sigma_u^2\}$ for $\forall \sigma_u^2 \in Q_+$, we get

$\mu_{\sigma_u^2}(Z_{\sigma_u^2}) = \mu_{\sigma_u^2}(\{x : f(x) = \sigma_u^2\}) = 1, \ \ \forall \sigma_u^2 \in Q_+$

$Z_{\sigma_{u_1}^2} \cap Z_{\sigma_{u_2}^2} = \{x : f(x) = \sigma_{u_1}^2\} \cap \{x : f(x) = \sigma_{u_2}^2\} = \varnothing, \ \ \forall \sigma_{u_1}^2 \neq \sigma_{u_2}^2$.

Hence, the statistical structure

$\left\{(-\infty, +\infty), B(-\infty, +\infty), \mu_{\sigma_u^2}, \sigma_u^2 \in Q_+\right\}$ is strongly separable.

Sufficiency. Since the statistical structure

$\left\{(-\infty, +\infty), B(-\infty, +\infty), \mu_{\sigma_u^2}, \sigma_u^2 \in Q_+\right\}$ is strongly separable, there exists a family $\{Z_{\sigma_u^2}, \sigma_u^2 \in Q_+\}$ of elements of the σ - algebra $B(-\infty, +\infty)$, such that

1) $\mu_{\sigma_u^2}(Z_{\sigma_u^2}) = 1, \ \forall \sigma_u^2 \in Q_+$;

2) $Z_{\sigma_{u_1}^2} \cap Z_{\sigma_{u_2}^2} = Q \ \ \forall \sigma_{u_1}^2 \neq \sigma_{u_2}^2, \ \sigma_{u_1}^2, \sigma_{u_2}^2 \in Q_+$.

Define the mapping $f : \{(-\infty, +\infty), B(-\infty, +\infty)\} \to [Q_+, B(Q_+)]$ as follows $f(Z_{\sigma_u^2}) = Z_{\sigma_u^2}$, $x \in Z_{\sigma_u^2}$. The mapping $f : \{(-\infty, +\infty), B(-\infty, +\infty)\} \to [Q_+, B(Q_+)]$ is a measurable mapping and $\mu_{\sigma_u^2}(\{x : f(x) = \sigma_u^2\}) = 1$, $\forall \sigma_u^2 \in Q_+$.

The following theorem is proven similarly to theorem 1.

Theorem 2. In order for Gaussian statistical structure $\left\{(-\infty,+\infty), B(-\infty,+\infty), \mu_{\sigma_u^2}, \sigma_u^2 \in Q_+\right\}$ to admit a consistent hypothesis testing criteria in the theory (ZFC) it is necessary and sufficient that this statistical structure be strongly separable (Definition 4).

Thus for strongly separable Gaussian statistical structures statistical structure estimators and consistent hypothesis testing criteria for σ_u^2 are constructed.

Because $y=\alpha+\beta x+u$, with $u \sim N(0,\sigma_u^2)$ and for σ_u^2 we have constructed consistent assessment and consistent criterion, we are left with α and β parameters, $Y \sim N(\alpha+\beta x,\sigma_u^2)$. We consider countable Gaussian statistical structures $\left\{(-\infty,+\infty), B(-\infty,+\infty), \mu_{\alpha,\beta}, \alpha\in\Theta, \beta\in\Theta\right\}$. The following theorems is proven similarly theorem 1 and theorem 2.

Theorem 3. In order for Gaussian statistical structure $\left\{(-\infty,+\infty), B(-\infty,+\infty), \mu_{\alpha,\beta}, \alpha\in\Theta, \beta\in\Theta\right\}$ to admit a consistent estimator of parameters $\alpha\in\Theta, \beta\in\Theta$ in the theory (ZFC), it is necessary and sufficient that this statistical structure be strongly separable (Definition 4).

Theorem 4. In order for Gaussian statistical structure $\left\{(-\infty,+\infty), B(-\infty,+\infty), \mu_{\alpha,\beta}, \alpha\in\Theta, \beta\in\Theta\right\}$ to admit a consistent hypothesis testing criteria in the theory (ZFC), it is necessary and sufficient that this statistical structure be strongly separable (Definition 4).

The Linear Regression Model for a continuum statistical structure.

Let $\{\mu_i, i\in I\}$ Card $I=C$ be probability measures defined on the measurable space (E,S). For each $i\in I$ denote by $\overline{\mu}_i$ the completion of the measure μ_i, and denote by $dom(\overline{\mu}_i)$ the σ- algebra of all $\overline{\mu}_i$-measurable subsets of E. Let $S_1=\bigcap_{i\in I} dom(\overline{\mu}_i)$.

Definition 9. The statistical structure $\{E,S_1,\overline{\mu}_i, i\in I\}$ is called strongly separable statistical structure if there exist the family S_1-measurable sets $\{Z_i, i\in I\}$ such that the following relations are fulfilled:

1) $\overline{\mu}_i(Z_i)=1$, $\forall i\in I$;
2) $Z_{i_1}\bigcap Z_{i_2}=\varnothing$, $\forall i_1\neq i_2$, $i_1,i_2\in I$;
3) $\bigcup_{i\in I} Z_i=E$.

Definition 10. We will say that the orthogonal statistical structure $\{E,S_1,\overline{\mu}_i, i\in I\}$ admits a consistent estimator of parameters $i\in I$ if there exists at least one measurable mapping

$$f:(E,S_1)\to(I,B(I)),$$

such that

$$\overline{\mu}_i(\{x: f(x)=i\})=1,\ \ \forall i\in I.$$

Definition 11. We will say that the orthogonal statistical structure $\{E,S_1,\overline{\mu}_h, h\in H\}$, Card $H=C$ admits a consistent criterion for hypothesis testing if there exists at least one measurable mapping

$$\delta:(E,S_1)\to(I,B(H)),$$

such that

$$\overline{\mu}_h(\{x:\delta(x)=h\})=1,\ \ \forall\sigma_h\in(0,+\infty).$$

Theorem 5. In order for Borel orthogonal Gaussian statistical structure $\{\{E, S_1, \overline{\mu}_{\sigma_u}, \sigma_u \in [0,+\infty)\}\}$ to admit "z-criterion" for hypothesis testing criteria in the theory (ZFC)§(MA), it is necessary and sufficient that this statistical structure be strongly separable (Definition 9).

Proof. Necessity. The existence of "z-criterion" for hypothesis testing $\delta:(E,S_1)\to(H,B(H))$ implies that $\overline{\mu}_{\sigma_H}(\{x:\delta(x)=\sigma_h\})=1,\ \forall h\in[0,+\infty)$.

Setting $X_{\sigma_H}=\{x:\sigma(x)=\sigma_H\}$ for $\sigma_u\in[0,+\infty)$ we get:

$\overline{\mu}_h(X_{\sigma_H})=1$, for $\forall\sigma_h\in[0,+\infty)$;

$X_{\sigma_{H'}}\cap X_{\sigma_{H'''}}=\varnothing$ for all different $\sigma_{H''}$ and $\sigma_{H''}$ from $[0,+\infty)$;

$\bigcup\limits_{\sigma_{H\in[0,\infty)}} X_{\sigma_H}=E.$

Hence the statistical structure
$\{\{E, S_1, \overline{\mu}_{\sigma_u}, \sigma_u \in [0,+\infty)\}\}$ is strongly separable.

Sufficiency. Since Gaussian statistical structure $\{E, S_1, \overline{\mu}_{\sigma_u}, \sigma_u \in [0,+\infty)\}$ is strongly separable, there exist family $\{Z_{\sigma_H}, \sigma_u\in[0,+\infty)\}$ of f elements of σ - algebra $S_1=\bigcap dom(\overline{\mu}_{\sigma_H})$, such that:

1) $\overline{\mu}_{\sigma_H}(Z_{\sigma_H})=1$, for $\forall\sigma_H\in[0,+\infty)$;

2) $Z_{\sigma_{H'}}\cap Z_{\sigma_{H'''}}=\varnothing$ for all different and from;

3) $\bigcup\limits_{\sigma_{H\in[0,\infty)}} Z_{\sigma_H}=E$. For $x\in E$. We put $\delta(x)=\sigma_u$ where σ_u is the unique hypothesis from the set $[0,+\infty)$ for which $x\in Z_{\sigma_{H'}}$ the existence of such a unique hypothesis from $[0,+\infty)$ can be proved using conditions 2), 3). Take now $Y\in B[0,+\infty)$, then $\{x:\delta(x)\in Y\}=\bigcup\limits_{\sigma_H\in Y} Z_{\sigma_H}$. If $\sigma_u\in Y$, then $\{x:\delta(x)\in Y\}=Z_{\sigma_{H_0}}\cup(\bigcup\limits_{\sigma_H\in Y-\sigma_{H_0}} Z_{\sigma_H})$. On the other hand, from the validity of conditions 1), 2), 3) it follows that $Z_{\sigma_{H_0}}\in S_1=\bigcap\limits_{\sigma_{H\in[0,\div\infty)}} dom(\overline{\mu}_{\sigma_H})\le dom(\overline{\mu}_{\sigma_{H_0}})$. On the other hand, the inclusion $\bigcup\limits_{\sigma_H\in Y-\sigma_{H_0}} Z_{\sigma_H}\subseteq E-Z_{\sigma_{H_0}}$ implies that $\overline{\mu}_{\sigma_{H_0}}(\bigcup\limits_{\sigma_H\in Y-\sigma_{H_0}} Z_{\sigma_H})=0$, and hence

$\bigcup\limits_{\sigma_H\in Y-\sigma_{H_0}} Z_{\sigma_H}\in dom(\overline{\mu}_{\sigma_{H_0}})$. Since $dom(\overline{\mu}_{\sigma_{H_0}})$ is a σ - algebra, we include that

$\{x:\delta(x)\in Y\}=Z_{\sigma_{H_0}}\cup(\bigcup\limits_{\sigma_H\in Y-\sigma_{H_0}} Z_{\sigma_H})\in dom(\overline{\mu}_{\sigma_{H_0}})$.

If $\sigma_{H_0}\notin Y$, then $\{x:\delta(x)\in Y\}=\bigcup\limits_{\sigma_H\in Y} Z_{\sigma_H}\subseteq E-Z_{\sigma_{H_0}}$ and we conclude that

$\overline{\mu}_{\sigma_{H_0}}(\{x:\delta(x)\in Y\})=0$. The last relation implies that

$\{x:\delta(x)\in Y\}\in dom(\overline{\mu}_{\sigma_{H_0}})$, $\forall Y\in B[0,+\infty)$. Hence

$\{x:\delta(x)\in Y\}\in\bigcap\limits_{\sigma_{H\in[0,\div\infty)}} dom(\overline{\mu}_{\sigma_H})=S_1$.

Therefore, the mapping $\delta:(E,S_1)\to([0,+\infty),B[0,+\infty))$ is a measurable mapping. Since $B[0,+\infty)$ contains all finite subsets of $[0,+\infty)$, we ascertain that $\overline{\mu}_{\sigma_H}(\{x:\delta(x)=\sigma_H\})=\overline{\mu}_{\sigma_H}(Z_{\sigma_H})=1$, $\forall\sigma_H\in[0,+\infty)$. i.e. this statistical structure "z-criterion" for hypothesis testing. The following theorems is proven similarly to theorem 5.

Theorem 6. In order for Borel orthogonal Gaussian statistical structure $\{\{E, S_1, \overline{\mu}_{\sigma_u}, \sigma_u \in [0,+\infty)\}\}$ to admit a consistent estimator of parameters $\sigma_H \in [0,+\infty)$, in the theory (ZFC)§(MA), it is necessary and sufficient that this statistical structure be strongly separable (Definition 9).

Theorem 7. In order for Borel orthogonal Gaussian statistical structure $\{\{E, S_1, \overline{\mu}_{\alpha.\beta}, \alpha \in R, \beta \in R\}\}$ to admit a consistent estimator of parameters $\alpha \in R,\ \beta \in R$ in the theory (ZFC)§(MA), it is necessary and sufficient that this statistical structure be strongly separable (Definition 9).

Theorem 8. In order for Borel orthogonal Gaussian statistical structure $\{\{E, S_1, \overline{\mu}_{\alpha.\beta}, \alpha \in R, \beta \in R\}\}$ to admit a consistent hypothesis testing in the theory (ZFC)§(MA), it is necessary and sufficient that this statistical structure be strongly separable (Definition 9).

მათემატიკა

წრფივი რეგრესიის კოეფიციენტების ჰიპოთეზათა შემოწმების ძალდებული კრიტერიუმები და პარამეტრის ძალდებული შეფასებები გაუსის სტატისტიკური სტრუქტურებისათვის

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§ აკაკი წერეთლის სახელმწიფო უნივერსიტეტი, ზუსტი და საბუნებისმეტყველო მეცნიერებების ფაკულტეტი, ქუთაისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

ნაშრომში ალბათური მახასიათებლების დასადგენად გამოყენებულია სტატისტიკური მეთოდი. სტატისტიკის პრობლემებს შორის არსებობს ამოცანების კლასი, რომელშიც დაკვირვების რაოდენობა უნიკალურია. ამ უნიკალურობის მიუხედავად, ბევრ შემთხვევაში შესაძლებელია უცნობი განაწილების პარამეტრების მნიშვნელობათა საიმედოდ განსაზღვრა ან განაწილების ზუსტი ფორმის შესახებ კონკურენტი ჰიპოთეზების უსასრულო რაოდენობისგან ერთ-ერთის საიმედოდ არჩევა. იმ შემთხვევაში, როდესაც პარამეტრი საიმედოდ განისაზღვრება ერთი დაკვირვებით, ნათქვამია, რომ მისთვის არსებობს პარამეტრის თანმიმდევრული შეფასება. ნაშრომში ჩვენ განვსაზღვრავთ გაუსის სტატისტიკურ სტრუქტურას, პარამეტრების თანმიმ-

დევრული შემფასებლების არსებობის აუცილებელ და საკმარის პირობებს და გაუსის სტატისტიკური სტრუქტურის თანმიმდევრული ჰიპოთეზების ტესტირების კრიტერიუმებს.

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