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Mathematics

Counting Points in a Real Semi-Algebraic Subset of the Plane

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Abstract. Effective computing of the number of points on a semi-algebraic subset given in explicit form is approached by a general algorithm. It turned out that an effective solution to this problem can be given using the signature method, which is based on computing the signatures of auxiliary quadratic forms. This method enables one to obtain an algorithm for finding the number of real roots of a system of polynomial equations within any domain defined by polynomial inequalities, thereby providing a complete solution to the problem of stable polynomials. *© 2026 Bull. Natl. Acad. Sci. Georg.*

Keywords: proper polynomial endomorphism, signature method, y-separable, semi-algebraic subset

Introduction

The present paper is dedicated to the problem of effective computing of the number of points of a semi-algebraic subset given in explicit form. This topic goes back to a problem of effective finding the number of roots of a real polynomial in the left half-plane, which was first posed by J. Maxwell. Under "effective finding" it is usually understood that one does not compute the roots themselves but indicates an algorithm which computes their number by means of a finite amount of algebraic and logical operations over the coefficients of the given polynomials.

Let $\varphi=(f,g):\mathbb{R}^2\to\mathbb{R}^2$ be a $\mathbb{C}-$ proper polynomial endomorphism of type (m,n) that is $deg\, f=n,\ deg\, g=m$ and $C_\infty(f_\mathbb{C})\cap C_\infty(g_\mathbb{C})=\varnothing$ ($C_\infty(f_\mathbb{C})$ and $C_\infty(g_\mathbb{C})$ are tangent cones at infinity of complexifications of f and g).

From properness it follows that there exist exactly $N=n\cdot m$ common complex roots. Denote them by $p_i=(\alpha_j,\beta_j),\ j=0,\dots,N-1.$

For the main construction, we must be guaranteed that all of these roots are simple and even have pairwise different ordinates, say, $\beta_i\neq\beta_j$ for any $i\neq j.$ Such mapping will be called y-precise. It was shown that it is possible to assume that everywhere without losing generality (see Propositions 1.1 and 1.2 and Lemma 1.2 from (Aliashvili, 2002)).

We also introduced the so-called "counting" quadratic form

$$Q_\chi^\varphi(\xi)=\sum_{j=0}^{N-1}\chi(\alpha_j,\beta_j)\left(\xi_0+\xi_1\beta_j+\xi_2\beta_j^2+\dots+\xi_{N-1}\beta_j^{N-1}\right)^2 \tag{1}$$

defined on an auxiliary N – dimensional Euclidean space $\mathbb{R}^N$ and depending on arbitrary real rational function (or polynomial) $\chi\in\mathbb{R}_2$, where $\mathbb{R}_2=\mathbb{R}[x,y]$ is the ring of real polynomials in two variables. If $\chi\equiv 1,$, we write $Q_1^\varphi\equiv Q^\varphi$ and call it the principal "counting" form (Krein & Neimark, 1981).

Theorem 1. If $\varphi=\mathbb{R}^2\to\mathbb{R}^2$ is an y-separable, $\mathbb{C}$ – proper polynomial endomorphism, then quadratic form Q^φ is nondegenerate and its signature $s(Q^\varphi)$ is equal number to the reals root of endomorphism φ

$$\#\varphi_{\mathbb{R}}^{-1}(0)=s(Q^\varphi). \tag{2}$$

One can also formulated an effective criterion of y-preciseness.

Proposition 1. If

$$R\left(R_x(f,g)(y),\right)R(f,g)'(y)\neq 0, \tag{3}$$

then $map\ \varphi=(f,g)$ is y-separable.

Here by $R_y(f,g)=R_y(f,g)(x)$ is denoted the resultant of two elements of ring $K[x][y]$. This is a polynomial of x.

Relying on the results of (Khimshiashvili, 2001) we can investigate more complicated case, where we have inequalities in addition to equations.

So, now suppose we are given a polynomial in two variables $h\in\mathbb{R}$, such that the triple (f,g,h) has no common roots: $Z_{\mathbb{C}}(f,g,h)=\varnothing$.

Analyzing the proof of Theorem 1 it is easy to notice that the introduction of a coefficient $h(\alpha_j,\beta_j)$ allows us to take into account roots belonging to the set $\{h>0\}$ as well. This enables us to get the main result giving a way to research zero-dimensional semi-algebraic subsets in $\mathbb{R}^2$.

Theorem 2. If $\varphi:\mathbb{R}^2\to\mathbb{R}^2$ is an y-separable, $\mathbb{C}$– proper polynomial endomorphism, $h\in\mathbb{R}_2$ and $Z_{\mathbb{C}}(f,g,h)=\varnothing,$, then quadratic form Q_h^φ is also non-degenerate and

$$\#\varphi_{\mathbb{R}}^{-1}(0)\cap\{h>0\}=\frac{1}{2}\left(s(Q^\varphi)+s(Q_h^\varphi)\right). \tag{4}$$

For $h\equiv 1$, we get Theorem 1.

Lemma 1. One has

$$(x+iy)(a+ib)^2+(x-iy)(a-ib)^2=\begin{cases}2x\left(a-\dfrac{yb}{x}\right)-2\dfrac{y^2+x^2}{x}b^2, & if\ x\neq 0\\ -y(a+b)^2+y(a-b)^2, & if\ x=0.\end{cases} \tag{5}$$

Proof of Theorem 2. We write the quadratic form Q_h^φ in a standard form

$$Q_h^{\varphi}(\xi) = \sum_{j=1}^{N-1} h(\alpha_j, \beta_j)\left(\xi_0 + \xi_1\beta_j + \xi_2\beta_j + \cdots + \xi_{N-1}\beta_j^{N-1}\right)^2. \tag{6}$$

Since (α_j, β_j) are common real roots of f and $g,$ and $Z_{\mathbb{C}}(f, g, h) = \varnothing$, then $h(\alpha_j, \beta_j) \neq \varnothing$ for all $j = 0,...,N-1.$ So some points of $\varphi_{\mathbb{R}}^{-1}(0)$ lie in the set $h(x,y) > 0,$ and the rest belong to the set $\{h(x,y) < 0\}.$

Considering the normal form of (6), similarly as in the previous Theorem 1 (cf. Aliashvili, 2002), we have k real roots and l pair complex-conjugate root, which we ennumerate as the mentioned theorem. Using the same change of variables, the form (6) will become

$$Q_h^{\varphi}(\eta) = \sum_{j=0}^{k-1} h(\alpha_j, \beta_j)\eta_j^2 + \sum_{j=1}^{l}\left[h(\alpha_{k+j-1}, \beta_{k+j-1})\eta_{k+j}^2 - h(\alpha_{k+l+j-1}, \beta_{k+l+j-1})\eta_{k+l+j-1}^2\right], \tag{7}$$

where $\eta_j, h(\alpha_j, \beta_j) \in \mathbb{R},$ for $1 \leq j \leq l,$ $\overline{\eta_{k+j-1}} = \beta_{k+l+j-1}$ and $h(\overline{\alpha_{k+j-1}}, \overline{\beta_{k+j-1}}) = h(\alpha_{k+l+j-1}, \beta_{k+l+j-1}),$ for $1 \leq j \leq l.$

It is obvious that this expression gives a nondegenerate quadratic form, as $h(\alpha_j, \beta_j) \neq 0$ for all $0 \leq j \leq N-1;$ hence the form is nondegenerate. Moreover, the determinant of this transform is a Vandermonde determinant, which can be easily calculated and equals to $\prod_{i \neq j}^{N}(\beta_i - \beta_j) \neq 0.$

So, Lemma 1 enables us to bring the second sum to the form, where the numbers of positive and negative squares are equal, i.e., they have no influence on the signature of the form Q_h^{φ} . Hence,

$$s(Q_h^{\varphi}) = \#\left[\varphi_{\mathbb{R}}^{-1}(0) \cap \{h > 0\}\right] - \#\left[\varphi_{\mathbb{R}}^{-1}(0) \cap \{h < 0\}\right].$$

On the other hand,

$$s(Q^{\varphi}) = \#\left[\varphi_{\mathbb{R}}^{-1}(0) \cap \{h > 0\}\right] - \#\left[\varphi_{\mathbb{R}}^{-1}(0) \cap \{h < 0\}\right].$$

Summing up these equalities yields formula (4) is obtained easily. The theorem is proved.

მათემატიკა

სიბრტყის ნახევრად ალგებრული სიმრავლის წერტილების რაოდენობის გამოთვლა

თ. ალიაშვილი

ილიას სახელმწიფო უნივერსიტეტი, ბიზნესის, ტექნოლოგიისა და განათლების ფაკულტეტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

ნახევრად ალგებრული ქვესიმრავლის წერტილების რაოდენობის ეფექტური გამოთვლა ცხადი სახით მიიღწევა ზოგადი ალგორითმით. აღმოჩნდა, რომ ამ პრობლემის ეფექტური გადაწყვეტა შესაძლებელია დამხმარე კვადრატული ფორმების სიგნატურის გამოთვლაზე დაფუძნებული სიგნატურული მეთოდის გამოყენებით. ეს მეთოდი საშუალებას იძლევა მივიღოთ ალგორითმი, პოლინომური უტოლობებით განსაზღვრულ ნებისმიერ არეში, პოლინომური განტოლებების სისტემის ნამდვილი ფესვების რაოდენობის მოსაძებნად, რაც იძლევა მდგრადი პოლინომების ამოცანის სრულ გადაწყვეტას.

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