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Mathematics

The M/G/1 Queueing System with an Unreliable Server: New Solution

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Abstract. This paper examines an M/G/1 queueing system with an unreliable server. The server experiences failures with a constant intensity, while its renewal time is a random variable with a general distribution. A semi-Markov model of the system is constructed using the method of supplementary variables. To study the model, an alternative approach is applied – one that does not rely on partial differential equations (Kolmogorov equations). The stationary (steady-state) regime is investigated in terms of operational calculus, specifically Laplace transforms and generating functions. *© 2026 Bull. Natl. Acad. Sci. Georg.*

Keywords: M/G/1 queueing system, semi-Markov process, Laplace transform

Introduction

For practical applications, it is important to study service systems that take into account the possibility that the server may break down and later be repaired. It is clear that there exist service systems in which, for a given arrival rate, the waiting time and queue length have an ergodic distribution. However, if we also consider the time spent repairing a regularly failing server, the queue will inevitably grow without bound. Therefore, it is essential to determine conditions for ergodicity and to obtain various performance characteristics for unreliable servers.

From a practical point of view, considerable interest lies in mathematical models that describe server failures and repair processes, as well as the service mechanism for arriving jobs when the server is not functioning. The server may fail during service, when idle, or in both situations. In each case, a statistical law governing the failure events must be specified. Often, real systems are well described by assuming that the server fails completely at random: the probability of failure in an interval of length h is $\alpha h + o(h)$, regardless of past history. This assumption is equivalent to the statement that the working time of the server in the idle state follows an exponential distribution (Gnedenko & Kovalenko, 1989).

In more general cases, the continuous working time of the server is a random variable whose distribution need not be exponential. In real service systems, one may be interested in a scheme in which the probability of failure in the interval $(t, t+h)$ depends on the duration the server has been continuously operating since the start of its current working period. Frequently, only the time elapsed since the last repair matters. The repair times are usually considered as independent, identically distributed random variables, though their distribution may differ depending on whether the failure occurred during service or while the server was idle. In many works, repair times are assumed exponential, but from a practical standpoint, it is important to drop this assumption.

Suppose a failure occurs during service. After the repair is completed, the job may return to the server to complete its service. In one scheme, the remaining service time does not depend on the progress already made, while in another, it diminishes according to the elapsed service time. Intermediate schemes are also possible. Another service mechanism of interest is one where the job may leave the system with a certain probability if the server fails during its service (Gnedenko & Kovalenko, 1989).

In this paper, however, we analyze the case where the server fails only when it is idle.

Analysis of Queue Length in the Steady State

Let $\{\Omega, F, \mathbb{P}\}$ be a probability space (Ω is sample space, F is σ-algebra of subsets of Ω and $\mathbb{P}$ is probability measure). All random events, random variables and random processes in this paper are defined in the $\{\Omega, F, \mathbb{P}\}$.

In an M/G/1 system, requests arrive according to a Poisson process with rate λ. The service time is a random variable with a general distribution function B. This is the service time when the server operates without experiencing failures $b(x) = B'(x)$; the service intensity is $\eta(x) = \dfrac{b(x)}{1-B(x)}$.

The server fails with intensity α. Its repair time is a random variable with a general distribution, $g(x) = G'(x)$; with repair intensity $\gamma(x) = \dfrac{g(x)}{1-G(x)}$.

We define the stochastic processes and functions $s(t)$, $n(t)$, $\xi(t)$, $P_0(t)$, $p_n(x,t)$, $q_n(x,t)$.

$s(t) = 0$, if at moment t the server is working; $s(t) = 1$, if at moment t the server is failed.

$n(t)$ is the number of requests in the system at moment t.

$\xi(t)$ is the time that has passed since the last 0-moment (the 0-moment is the moment a request service begins or the moment a failed server repair begins).

Let us denote

$$P_0(t) = \mathbb{P}\{n(t) = 0, s(t) = 0\},$$

$$p_n(x,t) = \lim_{h \to 0} \frac{1}{h} \mathbb{P}\{n(t) = n, 0 < \xi(t) < x+h, s(t) = 0\},$$

$$q_n(x,t) = \lim_{h \to 0} \frac{1}{h} \mathbb{P}\{n(t) = n, s(t) = 1, 0 < \xi(t) < x+h\}.$$

Using standard probabilistic reasoning, the following equations are obtained:

$$P_0'(t) = -(\lambda+\alpha) P_0(t) + \int_0^\infty p_1(x,t) b(x) dx, \tag{1}$$

$$\frac{\partial p_n(x,t)}{\partial t} + \frac{\partial p_n(x,t)}{\partial x} = -\lambda p_n(x,t) + \lambda p_{n-1}(x,t), \quad n \ge 1, \tag{2}$$

$$\frac{\partial q_0(x,t)}{\partial t}+\frac{\partial q_0(x,t)}{\partial x}=-(\lambda+\gamma(x))q_0(x,t), \tag{3}$$

$$\frac{\partial q_n(x,t)}{\partial t}+\frac{\partial q_n(x,t)}{\partial x}=-(\lambda+\gamma(x))q_n(x,t). \tag{4}$$

For boundary values, we obtain the following equations:

$$p_1(0,t)=p_0(t)\lambda+\int_0^t p_2(x,t)\eta(x)dx+\int_0^t q_1(x,t)\gamma(x)dx, \tag{5}$$

$$p_n(0,t)=\int_0^t p_{n+1}(x,t)\eta(x)dx+\int_0^t q_n(x,t)\gamma(x)dx.. \tag{6}$$

This system (1)-(6) together with initial and boundary (5) and (6) conditions constitute a nonclassical boundary value problem of mathematical physics with nonlocal boundary conditions.

Let us consider the stationary state of the system, which means that $t\to\infty$. Let us assume that stationary values exist for all functions. Namely,

$$\lim_{t\to\infty}P_0(t)=P_0,\quad \lim_{t\to\infty}p_n(x,t)=p_n(x),\quad n\geq 1,$$

$$\lim_{t\to\infty}q_n(x,t)=q_n(x),\quad n\geq 0,$$

$$\lim_{t\to\infty}p_n(0,t)=p_n(0),\quad n\geq 1.$$

We obtain from (1)-(6) equations:

$$(\lambda+\alpha)P_0=\int_0^\infty p_1(x)b(x)dx, \tag{7}$$

$$\frac{\partial p_1(x,t)}{\partial x}=-p_1(x)(\lambda+\alpha+\eta(x)), \tag{8}$$

$$\frac{\partial p_n(x,t)}{\partial x}=-\lambda p_n(x)+\lambda p_{n-1}(x),\quad n\geq 2, \tag{9}$$

$$\frac{\partial q_0(x)}{\partial x}=-(\lambda+\gamma(x))q_0(x), \tag{10}$$

$$\frac{\partial q_n(x)}{\partial x}=-(\lambda+\gamma(x))q_n(x), \tag{11}$$

$$p_1(0)=\lambda P_0+\int_0^\infty p_2(x)\eta(x)dx+\int_0^\infty q_1(x)\gamma(x)dx, \tag{12}$$

$$q_0(0)=\alpha P_0, \tag{13}$$

$$q_n(0)=0,\quad n\geq 1. \tag{14}$$

Theorem 1. The functions $p_n(x)$ and $q_n(x)$ have the form

$$P_n(0)=[1-B(x)]\sum_{k=1}^{n}P_k(0)\frac{(\lambda x)^{n-k}}{(n-k)!}e^{-\lambda x}, \tag{15$_1$}$$

$$q_n(0)=\left[1-G(x)\right]\sum_{k=0}^{n}q_k(0)\frac{(\lambda x)^{n-k}}{(n-k)!}e^{-\lambda x}. \tag{15_2}$$

The proof of this theorem is analogous of proofs given in (Khurodze & Kakubava 2025; Khurodze et al., 2025; Khurodze et al., 2024; Khurodze et al., 2023; Khurodze et al., 2022; Khurodze et al., 2020).

Let us introduce the generating functions. $L_1(z,x)$ and $L_2(z,x)$:

$$L_1(z,x)=\sum_{n=1}^{\infty}z^n p_n(x),$$

$$L_2(z,x)=\sum_{n=0}^{\infty}z^n q_n(x).$$

$$L(z,x)=L_1(z,x)+L_2(z,x).$$

Theorem 2. It is proved that

$$L_1(z,x)=L_1(z,0)\left[1-B(x)\right]e^{-\lambda(1-z)x}, \tag{16}$$

$$L_2(z,x)=L_2(z,0)\left[1-G(x)\right]e^{-\lambda(1-z)x}=\alpha P_0\left[1-G(x)\right]e^{-\lambda(1-z)x}. \tag{17}$$

The proof of this theorem is analogous of proofs given in (Khurodze & Kakubava 2025; Khurodze et al., 2025; Khurodze et al., 2024; Khurodze et al., 2023; Khurodze et al., 2022; Khurodze et al., 2020).

From (7)-(14) we obtain

$$(\lambda+\alpha)P_0 z+zL_1(z,0)=\int_0^{\infty}L_1(z,x)\eta(x)dx+\int_0^{\infty}L_2(z,x)\gamma(x)dx+\lambda P_0\cdot z^2. \tag{18}$$

Taking (7)-(14), (15_1) and (15_2) into account, we get:

$$(\lambda+\alpha)zP_0+zL_1(z,0)=\lambda z^2P_0+L_1(z,0)b^*\left(\lambda(1-z)\right)+\alpha P_0 g^*\left(\lambda(1-z)\right), \tag{19}$$

$$L_1(z)=\sum_{n=1}^{\infty}z^n p_n=\int_0^{\infty}L_1(z,x)dx=zP_0\frac{1-b^*(\lambda-\lambda z)}{b^*(\lambda-\lambda z)-z}. \tag{20}$$

Using L'Hôpital's rule, the following expression is obtained

$$L_2(1)=\alpha P_0\cdot\tau_1,$$

where τ is the average service time and τ_1 is the repair time. Let us denote $\lambda\tau=\rho$,

$$L_1(1)=\frac{\rho}{1-\rho}\cdot P_0.$$

The normalization condition takes the form:

$$P_0+L_1(1)+L_2(1)=1$$

from this expression we obtain

$$P_0=\frac{1-\lambda\tau}{1+\lambda\tau_1}.$$

From (16), (17) and the expression for P_0, we obtain the explicit form of the function $L_1(z,0)$. After this, using (16) and (17), we obtain explicit forms of the functions $L_1(z,x)$ and $L_2(z,x)$. Using P_0, (16) and (17) we obtain the expression for $L_1(z,x)$. Then we get the explicit form of $L(z,x)$. Finally, using the generating function $L(z,x)$, any numerical characteristic of the queue length (mathematical expectation, variance, etc.) can be found.

Conclusion

The paper employs a new and remarkably simple probabilistic approach to studying semi-Markov systems, showing that the classical use of partial differential equations in the boundary value problem (BVP) is unnecessary (Kakubava, 2020-2021). Although PDEs have long been considered the central and most important component of the BVP that arises due the use of supplementary variables technique, the results here demonstrate that all essential characteristics of considered system can be obtained using only the integro-differential equation together with the initial and non-local integral boundary conditions.

This insight reveals that, for semi-Markov models – including the classical Cox-type M/G/1 models with vacations, retrials, reneging, balking, finite waiting room, feedback, and others – the PDE component can be completely omitted without losing correctness. The approach also provides general forms for the corresponding PDE solutions, allowing anyone to verify consistency through direct substitution.

The authors suggest that future research may further develop this method through functional analysis, comparative and asymptotic studies, numerical techniques, and renewed investigation of well-posedness, which now arises under new conditions due to the absence of PDEs.

Remember, once again, that the supplementary variables method leads to a non-classical BVP of mathematical physics with nonlocal boundary conditions. The main parts of the BVP include the integro-differential equation (or a system of such equations); the partial-differential equation (PDE) (or a system of such equations); the Kolmogorov equations; the initial conditions; and the integral boundary equation (or a system of such equations).

We believe that our approach reveals the intrinsic probabilistic nature of SM processes, and it is impossible to discover this nature by studying the BVP without full comprehension of the considered systems.

This approach can be applied to any SM models, where the supplementary variables method is used, including all models based on of Cox's (1955) approach (M/G/1 with vacations, retrial, reneging, balking, finite waiting room, feedback and, etc.) also all semi-Markov models in the mathematical theory of reliability (MTR), etc.

The suggested alternative approach has the following advantages: 1) It provides a simple solution to the BVP; 2) It gives a general form to solution for PDE ofthis type, regardless of the context in which they arise- a result that should be of great interest to experts in the field of PDEs, which is why these equations are retained. 3) If a reader questions the consistency of the suggested approach, he can do so by direct substitution of the above general solutions into the corresponding equations.

We hope that leading experts in the field will further develop this approach in various directions, including: 1) functional analysis; 2) comparative studies; 3) asymptotic analysis; 4) well-posedness of mathematical models, 5) application of numerical methods, etc.

Regarding the well-posedness of the constructed models as an abstract Cauchy problem, our new method has significantly changed the problem due to expressions (15). Thus, the question of well-posedness arises again, but under new conditions. This issue remains open, and we encourage researchers wirh interest in the field to engage with it.

მათემატიკა

M/G/1 რიგის სისტემა არასაიმედო სერვერით – ახალი გადაწყვეტა

ნ. სალია

საქართველოს ტექნიკური უნივერსიტეტი, ინფორმატიკისა და მართვის სისტემების ფაკულტეტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ა. მესხის მიერ)

ნაშრომში განხილულია M/G/1 რიგის სისტემა არასაიმედო სერვერით. სერვერი განიცდის მტყუნებებს მუდმივი ინტენსივობით და მისი აღდგენის დრო არის შემთხვევითი სიდიდე ზოგადი განაწილებით. სისტემის ნახევრად მარკოვის მოდელი აგებულია დამატებითი ცვლადის გამოყენებით. მოდელის შესასწავლად გამოიყენება ალტერნატიული მიდგომა, რომელიც არ არის დაფუძნებული კერძოწარმოებულიან დიფერენციალურ განტოლებებზე (კოლმოგოროვის განტოლებები). გამოკვლეულია სისტემის სტაციონარული მდგომარეობა, ოპერაციული აღრიცხვის ტერმინებში (კერძოდ, ლაპლასის გარდაქმნებისა და მაწარმოებელი ფუნქციების გამოყენებით).

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