

*Mathematics*

# (Co)homology of $\Gamma$ -groups and $\Gamma$ -equivariant homology-I

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**Abstract.** This paper surveys the theory of  $\Gamma$ -equivariant homology and cohomology of  $\Gamma$ -groups and related structures, forming a framework known as  $\Gamma$ -homological algebra. We review the construction of equivariant chain complexes, the interpretation of extensions of  $\Gamma$ -groups in terms of cohomology, and the role of abstract kernels and obstruction theory. We also discuss rational computations for finite cyclic  $\Gamma$ -groups and outline the theory of  $\Gamma$ -equivariant Hochschild homology and derived functors. © 2026 Bull. Natl. Acad. Sci. Georg.

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## Introduction

The purpose of this paper is to present a survey of results on group actions in homological algebra, with a focus on  $\Gamma$ -equivariant homology and cohomology of  $\Gamma$ -groups and related algebraic structures. These constructions extend classical homological methods by incorporating compatible actions of a fixed group  $\Gamma$  on groups, rings, and chain complexes, leading to a framework that has become known as  $\Gamma$ -homological algebra.

The basic idea is to equip classical chain complexes computing homology of groups or rings with natural  $\Gamma$ -actions induced by the given actions on the underlying algebraic objects. Passing to homology then produces  $\Gamma$ -equivariant homology and cohomology groups, which generalize classical invariants and provide tools for studying extensions, derived functors, and related structures in an equivariant setting.

Equivariant approaches to extensions and homology have their origins in the work of Whitehead (Whitehead, 1950) and have subsequently appeared in various contexts in homotopy theory, algebraic  $K$ -theory, and the theory of crossed modules (Carlsson, 2001; Fiedorowicz et al., 1982; Kuku, 1984; Phillips, 1987). Over the past years a number of results have been obtained concerning  $\Gamma$ -equivariant extensions of groups, obstruction theory, homology of crossed  $\Gamma$ -modules, and equivariant Hochschild homology. The aim of this article is to bring these developments together and present them in a unified and streamlined form.

We review extensions of  $\Gamma$ -groups with  $\Gamma$ -section maps and their relation to equivariant cohomology, as well as extensions of crossed  $\Gamma$ -modules, which generalize relative extensions of group epimorphisms in the sense of Loday (Loday, 1978). We also discuss the construction of  $\Gamma$ -equivariant Hochschild homology and its connections with cyclic homology and equivariant group homology. Particular attention is given to examples and computations, including the rational  $\Gamma$ -equivariant (co)homology of finite cyclic  $\Gamma$ -groups.

**Notations.** Throughout the paper, the following notation will be used. For a  $\Gamma$ -group  $G$ ,  $[\Gamma G]$  denotes the set of elements  $\gamma g g^{-1}$  and  $\Gamma G$  the normal subgroup generated by it;  $G_\Gamma = G/\Gamma G$ . The subgroup  $[G, G]_\Gamma$  generated by the commutator subgroup and the elements  $\gamma g g^{-1}$  is called the  $\Gamma$ -commutator of  $G$ , and  $G_\Gamma^{ab}$  denotes the abelianization of  $G_\Gamma$ .

**Preliminaries.** In this section, we recall basic definitions and results from (Cegarra et al., 2002) which will be used later. We also indicate how equivariant analogues of classical homological properties of groups arise within this framework.

Let  $\mathbf{G}^\Gamma$  be the category whose objects are groups equipped with an action of a fixed group  $\Gamma$  (called  $\Gamma$ -groups) and whose morphisms are homomorphisms compatible with the  $\Gamma$ -action.

Any exact sequence of  $\Gamma$ -groups

$$1 \rightarrow A \rightarrow B \xrightarrow{\tau} G \rightarrow 1 \quad (1)$$

is called a  $\Gamma$ -extension of  $G$  by  $A$ . It is said to have a  $\Gamma$ -section if there exists a map  $\beta: G \rightarrow B$  such that  $\tau\beta = 1_G$  and  $\beta(\gamma g) = \gamma\beta(g)$ . If  $\beta$  is a homomorphism, the extension is split.

**Definition 1.** (i) A  $\Gamma$ -equivariant  $G$ -module  $A$  is a  $G$ -module equipped with a  $\Gamma$ -module structure such that

$$\sigma(\gamma a) = \sigma\gamma(\sigma a),$$

for all  $g \in G$ ,  $\sigma \in \Gamma$ , and  $a \in A$ . The category of  $\Gamma$ -equivariant  $G$ -modules is equivalent to the category of modules over the semidirect product  $G \rtimes \Gamma$  (Janelidze, 1982).

An extension with  $\Gamma$ -section map is called a  $\Gamma$ -equivariant extension if the kernel is a  $\Gamma$ -equivariant  $G$ -module.

(ii) A  $\Gamma$ -equivariant  $G$ -module is relatively free if it is a free  $G$ -module with basis a  $\Gamma$ -set, and relatively projective if it is a retract of a relatively free module.

The class  $\mathcal{P}$  of relatively projective  $\Gamma$ -equivariant  $G$ -modules forms a projective class with respect to proper sequences.

**Definition 2.** The  $\Gamma$ -equivariant homology and cohomology of a  $\Gamma$ -group  $G$  with coefficients in  $A$  are defined by

$$H_n^\Gamma(G, A) = \text{Tor}_n^{\mathcal{P}}(\mathbb{Z}, A), \quad H_\Gamma^n(G, A) = \text{Ext}_{\mathcal{P}}^n(\mathbb{Z}, A), \quad n \geq 0,$$

where tensor products and  $\text{Hom}$  are taken over  $\mathbb{Z}(G \rtimes \Gamma)$  and  $\mathbb{Z}$  has trivial  $G$  and  $\Gamma$  action.

These groups can be computed using the  $\Gamma$ -equivariant bar resolution of  $\mathbb{Z}$ . The action of  $\Gamma$  on  $G$  induces an action on the classical bar resolution, giving isomorphisms

$$H_n^\Gamma(G, A) \cong H_n(B_* \otimes_{G \rtimes \Gamma} A), \quad H_\Gamma^n(G, A) \cong H^n(\text{Hom}_{G \rtimes \Gamma}(B_*, A)).$$

An equivalent description of cohomology is obtained via  $\Gamma$ -cochains  $C_\Gamma^n(G, A)$  consisting of  $\Gamma$ -maps  $G^n \rightarrow A$ . The resulting cochain complex computes  $H_\Gamma^n(G, A)$ ; in particular  $\ker \delta^1 = \text{Der}_\Gamma(G, A)$ .

Two  $\Gamma$ -equivariant extensions are equivalent if there exists a morphism inducing the identity on  $G$  and  $A$ . Denote the set of equivalence classes by  $E_\Gamma^1(G, A)$ .

**Theorem 3.** There is a bijection

$$E_\Gamma^1(G, A) \cong H_\Gamma^2(G, A).$$

*Remark 4.* With the Baer sum,  $E_\Gamma^1(G, A)$  becomes an abelian group and the above bijection is an isomorphism. Higher-dimensional analogues are obtained using  $n$ -fold  $\Gamma$ -equivariant extensions.

**Definition 5.** A  $\Gamma$ -group is called  $\Gamma$ -free if it is a free group with basis a  $\Gamma$ -set.

Let  $\mathbb{F}$  be the projective class of  $\Gamma$ -free groups in  $G^\Gamma$ .

**Theorem 6.** For  $n \geq 2$  there are isomorphisms

$$H_n^\Gamma(G, A) \cong L_{n-1}^\mathbb{F}(I(G) \otimes_{G \rtimes \Gamma} A), \quad H_\Gamma^n(G, A) \cong R_{\mathbb{F}}^{n-1} \text{Der}_\Gamma(G, A).$$

We also recall several results on  $\Gamma$ -equivariant integral homology  $H_n^\Gamma(G)$ .

**Theorem 7.** (i)  $L_n^\mathbb{F}(G_\Gamma^{ab}) \cong H_{n+1}^\Gamma(G)$ .

(ii) There are exact sequences relating classical and  $\Gamma$ -equivariant homology, involving the functor

$$U(G) = [G, G]_\Gamma / [G, G].$$

**Theorem 8.** If  $\alpha: P \rightarrow G$  is a  $\Gamma$ -projective presentation, then

$$H_2^\Gamma(G) \cong (R \cap [P, P]_\Gamma) / [P, R]_\Gamma,$$

where  $R = \ker \alpha$ .

**Definition 9.** A  $\Gamma$ -subgroup  $L$  of a  $\Gamma$ -group  $G$  is a retract if there exists a  $\Gamma$ -homomorphism  $G \rightarrow L$  restricting to the identity on  $L$ .

**Theorem 10.** If  $L$  is a retract of a  $\Gamma$ -free group, then for any  $\Gamma$ -equivariant module  $A$ :

$$H_n^\Gamma(L, A) = 0, \quad H_\Gamma^n(L, A) = 0, \quad n > 1.$$

The Nielsen–Schreier theorem does not extend in general to  $\Gamma$ -free groups; nevertheless, various sufficient conditions ensure that  $\Gamma$ -subgroups of  $\Gamma$ -free groups are again  $\Gamma$ -free (Zavalo, 1964).

Finally, there is a connection with equivariant cohomology of spaces.

**Theorem 11.** Let  $G$  act properly on a space  $X$  with a compatible  $\Gamma$ -action. If  $X$  is acyclic with trivial  $\Gamma$ -action or is  $\Gamma$ -contractible, then

$$H_\Gamma^n(G, A) \cong H_\Gamma^n(X/G, A).$$

**Extensions of  $\Gamma$ -groups.** We introduce an internal property of  $\Gamma$ -group extensions possessing a  $\Gamma$ -section map that will be used throughout the paper.

**Definition 12.** Sequence (1) of  $\Gamma$ -groups is said to possess the  $\Gamma$ -property if the restriction of  $\tau$  to the subset  $[\Gamma B]$  of  $B$  is injective.

**Theorem 13.** Sequence (1) possesses the  $\Gamma$ -property if and only if it has a  $\Gamma$ -section map and  $\Gamma$ -acts trivially on  $\ker \tau$ .

**Corollary 14.** The sequence  $E_\Gamma$  possesses the  $\Gamma$ -property and every section map is a  $\Gamma$ -section map.

**Definition 15.** A  $\Gamma$ -group  $G$  is called  $\Gamma$ -perfect if  $G = [G, G]_\Gamma$ , or equivalently if  $H_1^\Gamma(G) = 0$  (Loday, 1978; Cegarra et al., 2002).

**Example 16.** Let  $F(G)$  be the  $\Gamma$ -free group generated by the  $\Gamma$ -group  $G$ . The natural short exact sequence

$$1 \rightarrow R \rightarrow F(G) \xrightarrow{\tau} G \rightarrow 1$$

admits a  $\Gamma$ -section map. The induced central  $\Gamma$ -equivariant extension

$$0 \rightarrow R/[F(G), R]_\Gamma \rightarrow F(G)/[F(G), R]_\Gamma \rightarrow G \rightarrow 1$$

has the  $\Gamma$ -property.

If  $G$  is  $\Gamma$ -perfect, one obtains the universal central  $\Gamma$ -equivariant extension

$$0 \rightarrow R \cap [F(G), F(G)]_\Gamma/[F(G), R]_\Gamma \rightarrow [F(G), F(G)]_\Gamma/[F(G), R]_\Gamma \rightarrow G \rightarrow 1,$$

whose kernel is isomorphic to  $H_2^\Gamma(G)$  (Cegarra et al., 2002).

We now consider non-abelian extensions. Let

$$1 \rightarrow J \rightarrow X \rightarrow G \rightarrow 1$$

be an extension of  $\Gamma$ -groups with the  $\Gamma$ -property. Then  $\Gamma$  acts trivially on  $J$ , and conjugation induces a  $\Gamma$ -homomorphism

$$\psi: G \rightarrow \text{Aut}(J)/\text{Inn}(J).$$

**Definition 17.** The triple  $(G, J, \psi)$  is called the abstract kernel of the extension.

**Theorem 18.** (i) For any abstract kernel  $(G, J, \psi)$  there exists an obstruction

$$\text{Obs}(G, J, \psi) \in H_1^3(G, C),$$

where  $C$  is the center of  $J$ . The kernel admits an extension if and only if the obstruction vanishes.

(ii) If an extension exists, equivalence classes of extensions with  $\Gamma$ -property are in bijection with

$$H_1^2(G, C).$$

The construction of extensions follows the classical case using factor systems and semi-direct product  $\Gamma$ -extensions; details are analogous and therefore omitted.

To extend the correspondence between extensions and cohomology to higher dimensions, we introduce  $n$ -fold  $\Gamma$ -equivariant extensions.

An  $n$ -fold  $\Gamma$ -equivariant extension of  $G$  by  $A$  is a long exact sequence

$$0 \rightarrow A \rightarrow B_1 \rightarrow \cdots \rightarrow B_n \rightarrow G \rightarrow 1,$$

in which the intermediate short sequences are proper sequences of  $\Gamma$ -equivariant modules and the last term is a  $\Gamma$ -equivariant extension.

Let  $E_\Gamma^n(G, A)$  denote the set of equivalence classes of such extensions; it becomes an abelian group via the Baer sum.

**Theorem 19.** For  $n \geq 1$  there is an isomorphism

$$E_\Gamma^n(G, A) \cong H_\Gamma^{n+1}(G, A).$$

The proof follows from the theory of relative derived functors and the universality of the functor  $Ext_\Gamma^*$  with respect to the projective class of proper sequences of  $\Gamma$ -equivariant modules; the argument parallels the classical case and is omitted.

**Some computations.** It is natural to ask for explicit computations of the  $\Gamma$ -equivariant (co)homology of groups introduced in (Cegarra et al., 2002). As noted earlier, these groups vanish in degrees  $n > 1$  for retracts of  $\Gamma$ -free groups. Here we consider the case of finite cyclic  $\Gamma$ -groups.

For the computation of the classical (co)homology of a finite cyclic group  $\mathbb{Z}_m$  with generator  $t$ , one uses the well-known free resolution

$$\cdots \xrightarrow{D} \mathbb{Z}(\mathbb{Z}_m) \xrightarrow{N} \mathbb{Z}(\mathbb{Z}_m) \xrightarrow{D} \mathbb{Z}(\mathbb{Z}_m) \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0,$$

where  $D = t - 1$  and  $N = 1 + t + \cdots + t^{m-1}$ .

If  $\Gamma$  acts nontrivially on  $\mathbb{Z}_m$ , this resolution is generally not compatible with the action and therefore is not suitable for computing  $\Gamma$ -equivariant (co)homology.

To obtain computable formulas, we work rationally. Let  $B_* \otimes \mathbb{Q} \rightarrow \mathbb{Q}$  be the bar resolution tensored with  $\mathbb{Q}$ .

**Definition 20.** The rational homology and cohomology of a group  $G$  with coefficients in a  $\mathbb{Z}(G)$ -module  $A$  are defined by

$$\begin{aligned} {}_{\mathbb{Q}}H_n(G, A) &= H_n\left((B_* \otimes \mathbb{Q}) \otimes_{\mathbb{Z}(G)} A\right), \\ {}_{\mathbb{Q}}H^n(G, A) &= H^n\left(\text{Hom}_{\mathbb{Z}(G)}(B_* \otimes \mathbb{Q}, A)\right). \end{aligned}$$

These groups are independent of the chosen projective resolution and satisfy

$${}_{\mathbb{Q}}H_n(G, A) \cong H_n(G, (A \otimes \mathbb{Q})), \quad {}_{\mathbb{Q}}H^n(G, A) = H^n(G, \text{Hom}(\mathbb{Q}, A)).$$

**Definition 21.** Let  $\Gamma$  act on  $G$  and trivially on  $\mathbb{Q}$ . The rational  $\Gamma$ -equivariant homology and cohomology of  $G$  with coefficients in a  $\Gamma$ -equivariant module  $A$  are defined by

$$\begin{aligned} {}_{\mathbb{Q}}H_n^\Gamma(G, A) &= H_n\left((B_* \otimes \mathbb{Q}) \otimes_{\mathbb{Z}(G \rtimes \Gamma)} A\right), \\ {}_{\mathbb{Q}}H_n^\Gamma(G, A) &= H^n\left(\text{Hom}_{\mathbb{Z}(G \rtimes \Gamma)}(B_* \otimes \mathbb{Q}, A)\right). \end{aligned}$$

Assume  $\Gamma$  acts nontrivially on  $\mathbb{Z}_m$ , so that  ${}^y t = t^k$  for some  $k$  coprime to  $m$ . Consider the sequence of free  $\mathbb{Q}(\mathbb{Z}_m)$ -modules

$$\cdots \xrightarrow{D} \mathbb{Q}(\mathbb{Z}_m) \xrightarrow{N} \mathbb{Q}(\mathbb{Z}_m) \xrightarrow{D} \mathbb{Q}(\mathbb{Z}_m) \xrightarrow{\varepsilon} \mathbb{Q} \rightarrow 0,$$

where

$$D = t^{m-1} + t^{m-2} + \dots + t - (m-1), \quad N = 1 + t + \dots + t^{m-1}.$$

One verifies that this sequence is exact, compatible with the  $\Gamma$ -action, and consists of relatively projective  $\Gamma$ -equivariant modules. Hence it is a projective resolution of  $\mathbb{Q}$  in the category of  $\Gamma$ -equivariant modules.

Applying  $-\otimes_{\mathbb{Z}(\mathbb{Z}_m \rtimes \Gamma)} A$  gives a chain complex computing the rational  $\Gamma$ -equivariant homology of  $\mathbb{Z}_m$ .

**Theorem 22.** Let  $\Gamma$  act nontrivially on the finite cyclic group  $\mathbb{Z}_m$ . Then for any  $\Gamma$ -equivariant  $\mathbb{Z}(\mathbb{Z}_m \rtimes \Gamma)$ -module  $A$ ,

$$\begin{aligned} \mathbb{Q}H_0^\Gamma(\mathbb{Z}_m, A) &= \mathbb{Q} \otimes A_\Gamma, \\ \mathbb{Q}H_{2n-1}^\Gamma(\mathbb{Z}_m, A) &= \ker N_* / \text{Im } D_*, \\ \mathbb{Q}H_{2n}^\Gamma(\mathbb{Z}_m, A) &= \ker D_* / \text{Im } N_*, \quad n > 0. \end{aligned}$$

where  $D_*$  and  $N_*$  are induced by  $D$  and  $N$ .

For cohomology we apply  $\text{Hom}_{\mathbb{Z}(\mathbb{Z}_m \rtimes \Gamma)}(-, A)$  to the resolution above.

**Theorem 23.** Let  $\Gamma$  act nontrivially on  $\mathbb{Z}_m$ . Then, for any  $\Gamma$ -equivariant module  $A$ ,

$$\begin{aligned} \mathbb{Q}H_\Gamma^0(\mathbb{Z}_m, A) &= \text{Hom}(\mathbb{Q}, A^\Gamma), \\ \mathbb{Q}H_\Gamma^{2n-1}(\mathbb{Z}_m, A) &= \ker N^* / \text{Im } D^*, \\ \mathbb{Q}H_\Gamma^{2n}(\mathbb{Z}_m, A) &= \ker D^* / \text{Im } N^*, \quad n > 0. \end{aligned}$$

Thus, the rational  $\Gamma$ -equivariant (co)homology of finite cyclic  $\Gamma$ -groups is periodic, with period two in positive degrees.

**$\Gamma$ -derived functors and  $\Gamma$ -equivariant Hochschild homology.** Let  $\Gamma$  be a group acting on a unital ring  $\Lambda$ . A left  $\Lambda$ -module  $A$  equipped with an action of  $\Gamma$  satisfying

$$\gamma(\lambda a) = \gamma\lambda \gamma a$$

is called a  $\Gamma$ -equivariant left  $\Lambda$ -module.

Denote by  $\Gamma A$  the  $\Lambda$ -submodule generated by elements  $\gamma a - a$ , and by  $A_\Gamma$  the quotient  $A/\Gamma A$ .

**Definition 24.** A group  $\Gamma$  is said to act on a chain complex

$$\dots \rightarrow L_n \xrightarrow{\delta_n} L_{n-1} \rightarrow \dots$$

of  $\Lambda$ -modules if each  $L_n$  is a  $\Gamma$ -equivariant module and

$$\delta_n(\gamma l - l) \in \Gamma L_{n-1}.$$

The homology of the quotient complex  $L_*^\Gamma = (L_*)_\Gamma$  is called the  $\Gamma$ -equivariant homology of  $L_*$  and is denoted  $H_n^\Gamma(L_*)$ .

**Relation with cyclic homology.** Let  $A$  be a unital  $\kappa$ -algebra and  $M$  a  $\Gamma$ -equivariant  $A$ -bimodule. The Hochschild complex

$$C_*(A, M) = \dots \rightarrow M \otimes A^{\otimes n} \rightarrow \dots \rightarrow M$$

inherits a natural  $\Gamma$ -action.

**Definition 25.** The homology  $H_*^\Gamma(C_*(A, M))$  is called the  $\Gamma$ -equivariant Hochschild homology of  $A$  with coefficients in  $M$ , denoted  $H_*^\Gamma(A, M)$ . For  $M = A$  we write  $HH_*^\Gamma(A)$ .

When the cyclic group  $\mathbb{Z}$  acts by cyclic permutations on  $C_*(A, A)$ , the quotient complex coincides with the Connes complex. Hence cyclic homology over a field containing  $\mathbb{Q}$  appears as a special case of  $\Gamma$ -equivariant Hochschild homology.

**Relation with  $\Gamma$ -equivariant homology of groups.** Let  $G$  be a  $\Gamma$ -group and  $C_*(G)$  the standard chain complex computing integral homology. The induced action of  $\Gamma$  on generators gives a chain complex whose  $\Gamma$ -equivariant homology coincides with  $H_n^\Gamma(G)$ .

In particular, when  $A = \mathbb{Z}(G)$  with the induced action of  $\Gamma$ , one has natural isomorphisms

$$H_n^\Gamma(C_*(A, A)) \cong H_n^\Gamma(G).$$

Thus  $\Gamma$ -equivariant Hochschild homology contains both cyclic homology and  $\Gamma$ -equivariant group homology as special cases.

**$\Gamma$ -derived functors.** Let  $\mathbb{A}_\Lambda^\Gamma$  denote the category of  $\Gamma$ -equivariant left  $\Lambda$ -modules. A relatively free module is a free  $\Lambda$ -module with basis a  $\Gamma$ -set; a retract of such a module is called relatively projective.

A  $\Gamma$ -projective resolution of  $M$  is a complex

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

with each  $P_n$  relatively projective and all short exact sequences admitting  $\Gamma$ -sections.

**Definition 26.** For an additive functor  $T: \mathbb{A}_\Lambda^\Gamma \rightarrow \mathbb{A}_\mathbb{Z}^\Gamma$ , the left  $\Gamma$ -derived functors are defined by

$$L_n^\Gamma T(M) = H_n^\Gamma(TP_*(M)).$$

In particular, derived functors of the tensor product yield functors  $Tor_n^{\Lambda, \Gamma}(-, L)$ , recovering  $\Gamma$ -equivariant group homology in the case  $\Lambda = \mathbb{Z}(G)$ .

**$\Gamma$ -equivariant Hochschild homology.** For a  $\kappa$ -algebra  $A$  with  $\Gamma$ -action, define the  $\Gamma$ -additive commutator

$$[A, A]_\Gamma = \langle \sum a - a, aa' - a'a \rangle.$$

If  $A$  is commutative, let  $\Omega^1(A)$  be the module of Kähler differentials with the induced  $\Gamma$ -action, and define

$$\Omega_\Gamma^1(A) = (\Omega^1(A))_\Gamma.$$

**Theorem 27.** Let  $\Gamma$  act on a unital  $\kappa$ -algebra  $A$  and an  $A$ -bimodule  $M$ . Then:

- (a)  $HH_0^\Gamma(A) = A/[A, A]_\Gamma$ ;
- (b) If  $A$  is commutative, then  $HH_1^\Gamma(A) \cong \Omega_\Gamma^1(A)$ ;
- (c) If  $A$  is relatively projective as a  $\Gamma$ -equivariant  $\kappa$ -module, then

$$H_n^\Gamma(A, M) \cong Tor_n^{\Lambda, \Gamma}(A, M);$$

- (d) (Morita invariance) For all  $r \geq 1$ , then

$$H_n^\Gamma(A, M) \cong H_n^\Gamma(M_r(A), M_r(M)).$$

**Right derived functors.** Right derived functors  $Ext_{\Lambda, \Gamma}^n$  of  $Hom_\Lambda^\Gamma(-, M)$  are defined analogously. In particular, when  $\Lambda = \mathbb{Z}(G)$  one recovers

$$\text{Ext}_{\mathbb{Z}(G),\Gamma}^n(\mathbb{Z}, L) \cong H_{\Gamma}^n(G, L).$$

*Remark 28.* If  $\Gamma$  acts on a  $\kappa$ -algebra  $A$ , the combined action of  $\mathbb{Z} \times \Gamma$  on the Hochschild complex yields a notion of  $\Gamma$ -equivariant cyclic homology when  $\mathbb{Q} \subset \kappa$ .

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*მათემატიკა*

## $\Gamma$ -ჯგუფების (კო)ჰომოლოგია და $\Gamma$ -ეკვივარიანტული ჰომოლოგია-I

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ნაშრომში მოცემულია  $\Gamma$ -ჯგუფების  $\Gamma$ -ეკვივარიანტული ჰომოლოგიისა და კოჰომოლოგიის თეორია და მათთან დაკავშირებული სტრუქტურები, რომლებიც ერთიანდება  $\Gamma$ -ჰომოლოგიური ალგებრის ჩარჩოში. განვიხილავთ ეკვივარიანტული ჯაჭვური კომპლექსების აგებას,  $\Gamma$ -ჯგუფების გაფართოებების ინტერპრეტაციას კოჰომოლოგიის ტერმინებში, აგრეთვე აბსტრაქტული ბირთვების და დაბრკოლებათა თეორიის როლს. ასევე განვიხილავთ სასრულ ციკლურ  $\Gamma$ -ჯგუფებისთვის რაციონალურ გამოთვლებს და წარმოვაჩინოთ  $\Gamma$ -ეკვივარიანტული ჰომოლოგიის და წარმოებული ფუნქტორების თეორიას.

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