

Investigation of High-Order Time Differential Three-Phase-Lag Models of Thermoelastic Solids

Gia Avalishvili*, Mariam Avalishvili**

* Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University, Georgia

** School of Science and Technology, University of Georgia, Tbilisi, Georgia

(Presented by Academy Member Elizbar Nadaraya)

Abstract. In this paper, a high-order time differential approximation of the three-phase-lag dynamic three-dimensional model of thermoelastic solids is considered. The initial-boundary value problem with mixed boundary conditions corresponding to the nonclassical model with three phase-lag parameters is studied, where on certain parts of the boundary displacement and temperature vanish, and on the corresponding remaining parts of the boundary the densities of surface force and heat flux along the outward normal vector are given. A variational formulation of the initial-boundary value problem is obtained, and the existence and uniqueness result as well as continuous dependence of the solution on the given data in suitable spaces of vector-valued distributions are proved. © 2026 Bull. Georg. Natl. Acad. Sci.

Keywords: phase-lag thermoelasticity, initial-boundary value problem, existence, uniqueness, continuous

Introduction

The three-phase-lag model was proposed by S.K. Roy Choudhuri (2007) using the nonclassical models of thermoelastic bodies constructed by A.E. Green and P.M. Naghdi (1991) that depend not only on temperature but also on thermal displacement. Roy Choudhuri introduced three phase lags for the heat flux vector, the temperature gradient, and the thermal displacement gradient, and, instead of the classical Fourier law, proposed a time-delayed constitutive equation, which is a generalisation of the Tzou dual-phase-lag model (Tzou, 1995). In Dreher et al. (2009), it was proved that the delay equation corresponding to the Roy Choudhuri law of heat conduction is ill-posed. Hence, Taylor series approximations of this law are used instead of the Fourier law in the energy balance equation. In the paper (Roy Choudhuri, 2007), Roy Choudhuri considered two particular approximations of the three-phase-lag model and constructed the corresponding models for thermoelastic bodies. The well-posedness and exponential stability results were obtained for particular and some high-order approximations of Roy Choudhuri's three-phase-lag heat conduction equation in (Borgmeyer et al., 2014). The compatibility of a particular approximation of Roy Choudhuri heat conduction model with thermodynamical principles and continuous dependence results in

spaces of sufficiently smooth functions was studied by Chiriță et al. (2016). For one approximation of the Roy Choudhuri nonclassical model with three phase-lags for thermoelastic bodies, the domain of influence result was obtained in (Zampoli & Landi, 2017). The uniqueness and continuous dependence results in the spaces of classical smooth enough functions for the heat conduction equation with the high-order approximations of Roy Choudhuri's three-phase-lag model were proved in (Chiriță, 2018). The initial-boundary value problem corresponding to one of Roy Choudhuri's three-dimensional models of thermoelastic solids consisting of anisotropic homogeneous material with general mixed boundary conditions was studied by Avalishvili & Avalishvili (2023) and an approximation algorithm of the three-dimensional problem for thermoelastic plates by two-dimensional problems was constructed and investigated in (Avalishvili et al., 2024). Numerous publications have been devoted to the study of thermoelastic waves, stability of solutions, numerical and analytical methods of solution and related topics for the three-phase-lag models of thermoelastic solids (see Quintanilla & Racke, 2008; Kumar & Mukhopadhyay, 2010; Shakeriaski et al., 2021; Chiriță & Ciarletta, 2025 and the references given therein).

The present paper is devoted to the investigation of the nonclassical model of thermoelastic solids with three phase-lags, which is a generalisation of the Roy Choudhuri model (Roy Choudhuri, 2007), where the classical Fourier law of heat conduction is replaced by an approximation of Roy Choudhuri's three-phase-lag heat conduction law in which the lagged heat flux, temperature and thermal displacement gradients are substituted with high degree partial sums of the corresponding Taylor expansions with respect to the time variable. We consider the linear three-dimensional initial-boundary value problem corresponding to the nonclassical model of thermoelastic solids, with general mixed boundary conditions, where, on certain parts of the boundary, the densities of surface force and heat flux along the outward normal vector are given, and, on the corresponding remaining parts of the boundary, the displacement and temperature vanish. We obtain integral relations, which are equivalent to the original differential equations in the spaces of smooth enough functions, and, based on the integral relations, we consider the variational formulation in suitable spaces of vector-valued distributions with respect to the time variable with values in the Sobolev spaces of the three-dimensional initial-boundary value problem. Finally, we investigate the existence, uniqueness and continuous dependence of the solution on the given data in suitable function spaces.

Well-posedness of the time differential three-phase-lag model. We denote by $W^{r,2}(D) = H^r(D)$ and $H^{\hat{r}}(\hat{\Gamma})$, $r, \hat{r} \in \mathbf{R}$, $r \geq 0$, $0 \leq \hat{r} \leq 1$, the Sobolev spaces of orders $r, \hat{r}$ based on the spaces $H^0(D) = L^2(D)$ and $H^0(\hat{\Gamma}) = L^2(\hat{\Gamma})$ of square-integrable functions, respectively, where $D \subset \mathbf{R}^p$, $p \in \mathbf{N}$, is a bounded Lipschitz domain (McLean, 2000) and $\hat{\Gamma} \subset \partial D$ is a Lipschitz surface. We denote by $\mathbf{H}^r(D) = [H^r(D)]^3$, $\mathbf{H}^{\hat{r}}(\hat{\Gamma}) = [H^{\hat{r}}(\hat{\Gamma})]^3$, $\mathbf{L}^2(D) = [L^2(D)]^3$, $\mathbf{L}^{\hat{s}}(\hat{\Gamma}) = [L^{\hat{s}}(\hat{\Gamma})]^3$, $r \geq 0$, $0 \leq \hat{r} \leq 1$, $s \geq 1$, $r, \hat{r}, s \in \mathbf{R}$, the corresponding spaces of vector-valued functions. The trace operators are denoted by $tr_{\hat{\Gamma}} : H^1(D) \rightarrow H^{1/2}(\hat{\Gamma})$ and $tr_{\hat{\Gamma}} : \mathbf{H}^1(D) \rightarrow \mathbf{H}^{1/2}(\hat{\Gamma})$. For any $m \in \mathbf{N} \cup \{0\}$, we denote by $C^{m,1}(\bar{D})$ the space of functions whose all partial derivatives up to the order m are Lipschitz continuous on $\bar{D}$ (Ciarlet, 2013). For a Banach space X , we denote by $C([0, T]; X)$ the space of continuous functions on $[0, T]$ with values in X . $L^q(0, T; X)$, $1 \leq q \leq \infty$, is the space of such measurable functions $g : (0, T) \rightarrow X$ that $\|g(t)\|_X \in L^q(0, T)$ and the generalised first, second and arbitrary k -th order derivatives of g are denoted by $g' = dg/dt$, $g'' = d^2g/dt^2$ and $g^{(k)} = d^k g/dt^k$, $k \in \mathbf{N} \cup \{0\}$, $g^{(0)} = g$ (Dautray & Lions, 2000).

Let us consider a thermoelastic body with an initial configuration $\bar{\Omega} \subset \mathbf{R}^3$, which consists of an anisotropic inhomogeneous thermoelastic material. The body is clamped along a part Γ_0 of the Lipschitz boundary $\Gamma = \partial\Omega$ and, on the remaining part $\Gamma_1 = \Gamma \setminus \Gamma_0$, a surface force with density $\mathbf{g} = (g_i): \Gamma_1 \times (0, T) \rightarrow \mathbf{R}^3$ is given, where $\Gamma = \Gamma_0 \cup \Gamma_1$ is a Lipschitz dissection (McLean, 2000) of Γ ; the temperature θ vanishes along a part $\Gamma_0^\theta \subset \Gamma$ of the boundary and the heat flux along the outward normal vector of Γ , with density $\tilde{g}^\theta: \Gamma_1^\theta \times (0, T) \rightarrow \mathbf{R}$, is given on the remaining part $\Gamma_1^\theta = \Gamma \setminus \Gamma_0^\theta$ of the boundary, where $\Gamma = \Gamma_0^\theta \cup \Gamma_1^\theta$ is a Lipschitz dissection of Γ . The body is subjected to an applied body force with density $\mathbf{f} = (f_i): \Omega \times (0, T) \rightarrow \mathbf{R}^3$ and a heat source with density $\tilde{f}^\theta: \Omega \times (0, T) \rightarrow \mathbf{R}$. The nonclassical dynamic linear three-dimensional model of stress-strain state of the thermoelastic body $\bar{\Omega}$, in which the classical Fourier law of heat conduction is replaced with the high-order approximation

$$\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^r q_i(x, t)}{\partial t^r} = - \sum_{j=1}^3 \lambda_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_1} \frac{\tau_1^s}{s!} \frac{\partial^s \theta(x, t)}{\partial t^s} \right) - \sum_{j=1}^3 \bar{\lambda}_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_2} \frac{\tau_2^s}{s!} \frac{\partial^s \theta^d(x, t)}{\partial t^s} \right), \quad i = 1, 2, 3,$$

of the three-phase-lag law of heat conduction proposed by Roy Choudhuri (Roy Choudhuri, 2007), where $\tilde{q} = (q_i)_{i=1}^3$ is the heat flux vector, $\theta^d: \bar{\Omega} \times [0, T] \rightarrow \mathbf{R}$ is the thermal displacement such that $\partial \theta^d / \partial t = \theta$ and $\theta: \bar{\Omega} \times [0, T] \rightarrow \mathbf{R}$ is the temperature distribution, is given by the following initial-boundary value problem in differential form:

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \sum_{j=1}^3 \frac{\partial}{\partial x_j} \left(\sum_{p,q=1}^3 \mu_{ijpq} e_{pq}(\mathbf{u}) + \eta_{ij} \theta \right) = f_i \quad \text{in } \Omega \times (0, T), \quad i = 1, 2, 3, \quad (1)$$

$$\begin{aligned} \chi \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+2} \theta}{\partial t^{r+2}} - \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} \left(\lambda_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_1} \frac{\tau_1^s}{s!} \frac{\partial^{s+1} \theta}{\partial t^{s+1}} \right) \right) - \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} \left(\bar{\lambda}_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_2} \frac{\tau_2^s}{s!} \frac{\partial^s \theta}{\partial t^s} \right) \right) - \\ - \Theta_0 \sum_{i,j=1}^3 \eta_{ij} e_{ij} \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+2} \mathbf{u}}{\partial t^{r+2}} \right) = \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \tilde{f}^\theta}{\partial t^{r+1}} \quad \text{in } \Omega \times (0, T), \end{aligned} \quad (2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0 \times (0, T), \quad \sum_{j=1}^3 \left(\sum_{p,q=1}^3 \mu_{ijpq} e_{pq}(\mathbf{u}) + \eta_{ij} \theta \right) \nu_j = g_i \quad \text{on } \Gamma_1 \times (0, T), \quad i = 1, 2, 3, \quad (3)$$

$$\begin{aligned} \theta = 0 \quad \text{on } \Gamma_0^\theta \times (0, T), \\ - \sum_{i,j=1}^3 \lambda_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_1} \frac{\tau_1^s}{s!} \frac{\partial^{s+1} \theta}{\partial t^{s+1}} \right) \nu_i - \sum_{i,j=1}^3 \bar{\lambda}_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_2} \frac{\tau_2^s}{s!} \frac{\partial^s \theta}{\partial t^s} \right) \nu_i = \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \tilde{g}^\theta}{\partial t^{r+1}} \quad \text{on } \Gamma_1^\theta \times (0, T), \end{aligned} \quad (4)$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \frac{\partial \mathbf{u}}{\partial t}(x, 0) = \mathbf{u}_1(x), \quad x \in \Omega, \quad (5)$$

$$\theta(x, 0) = \theta_0(x), \quad \frac{\partial \theta}{\partial t}(x, 0) = \theta_1(x), \quad \dots, \quad \frac{\partial^{n_0+1} \theta}{\partial t^{n_0+1}}(x, 0) = \theta_{n_0+1}(x),$$

where $e_{ij}(\mathbf{v}) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$, $i, j = 1, 2, 3$, $\mathbf{v} = (v_i)_{i=1}^3: \Omega \rightarrow \mathbf{R}^3$, is the linearized strain tensor, $\mathbf{v} = (v_i)_{i=1}^3$

is the unit outward normal to Γ , $\mathbf{u} = (u_i)_{i=1}^3: \bar{\Omega} \times [0, T] \rightarrow \mathbf{R}^3$ is the displacement vector-function of the thermoelastic body, $\mathbf{u}_0 = (u_{0i})_{i=1}^3$ and $\mathbf{u}_1 = (u_{1i})_{i=1}^3$ are the initial displacement and velocity vector-functions, and θ_0 is the initial distribution of temperature, θ_1 is the rate of change of temperature, and θ_k , $k = 2, \dots, n_0 + 1$, are the k -th order derivatives of temperature with respect to the time variable at the initial moment, ρ is the mass density in the reference configuration, $(\mu_{ijpq})_{i,j,p,q=1}^3$ is the elasticity tensor, $(\eta_{ij})_{i,j=1}^3$ is the stress-temperature tensor, $\chi > 0$ is the volumetric heat capacity, $(\lambda_{ij})_{i,j=1}^3$ is the thermal conductivity

tensor and $(\bar{\lambda}_{pq})_{p,q=1}^3$ is the Green-Naghdi thermal conductivity rate tensor included in the Green-Naghdi models (Green & Naghdi, 1991) to consider the dependence on thermal displacement gradient, $\Theta_0 > 0$ is the temperature of the thermoelastic body in the natural state of no deformation, which is considered as a reference temperature, $\tau_0 \geq 0$, $\tau_1 \geq 0$ and $\tau_2 \geq 0$ stand for the heat flux, temperature gradient and thermal displacement gradient phase-lags, respectively. Note that the considered model is a generalisation of the Roy Choudhuri three-phase-lag model and coincides with it in the case of $n_0 = 2$ and $n_1 = n_2 = 1$ (Roy Choudhuri, 2007; Avalishvili & Avalishvili, 2023), the Lord-Shulman model (Lord & Shulman, 1967; Avalishvili & Avalishvili, 2014) is a particular case of this model for $n_0 = 1$, $n_1 = 0$ and $\bar{\lambda}_{pq} = 0$, $p, q = 1, 2, 3$, the high-order time differential dual-phase-lag models (Avalishvili & Avalishvili, 2025) are particular cases of this model for $\bar{\lambda}_{pq} = 0$, $p, q = 1, 2, 3$, and in the case of $\tau_0 = \tau_1 = 0$ and $\bar{\lambda}_{pq} = 0$, $p, q = 1, 2, 3$, the nonclassical three-phase-lag model (1)-(5) reduces to the classical linear three-dimensional model of thermoelastic solids.

We assume that the elasticity tensor $(\mu_{ijpq})_{i,j,p,q=1}^3$, the stress-temperature tensor $(\eta_{ij})_{i,j=1}^3$, the thermal conductivity tensor $(\lambda_{ij})_{i,j=1}^3$ and the thermal conductivity rate tensor $(\bar{\lambda}_{pq})_{p,q=1}^3$ satisfy the following symmetry conditions:

$$\mu_{ijpq} = \mu_{ijqp} = \mu_{jipq}, \quad \eta_{ij} = \eta_{ji}, \quad \lambda_{pq} = \lambda_{qp}, \quad \bar{\lambda}_{pq} = \bar{\lambda}_{qp}, \quad i, j, p, q = 1, 2, 3. \quad (6)$$

If $\mathbf{u} = (u_i)_{i=1}^3$ and θ are smooth enough, then by multiplying equations (1) by arbitrary continuously differentiable functions $v_i : \bar{\Omega} \rightarrow \mathbf{R}$ ($i = 1, 2, 3$), which vanish on Γ_0 , and equation (2) by a continuously differentiable function $\varphi : \bar{\Omega} \rightarrow \mathbf{R}$, such that $\varphi = 0$ on Γ_0^θ , by integrating on Ω , by using the Green identity (McLean, 2000), and taking into account the symmetry properties (6) and the boundary conditions (3), (4), we obtain the following integral relations:

$$\sum_{i=1}^3 \int_{\Omega} \rho \frac{\partial^2 u_i}{\partial t^2} v_i dx + \sum_{i,j,p,q=1}^3 \int_{\Omega} \mu_{ijpq} e_{pq}(\mathbf{u}) e_{ij}(\mathbf{v}) dx + \sum_{i,j=1}^3 \int_{\Omega} \eta_{ij} \theta e_{ij}(\mathbf{v}) dx = \sum_{i=1}^3 \int_{\Omega} f_i v_i dx + \sum_{i=1}^3 \int_{\Gamma_1} g_i v_i d\Gamma, \quad (7)$$

$$\begin{aligned} & \int_{\Omega} \chi \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \theta}{\partial t^{r+1}} \varphi dx + \sum_{i,j=1}^3 \int_{\Omega} \lambda_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_1} \frac{\tau_1^s}{s!} \frac{\partial^{s+1} \theta}{\partial t^{s+1}} \right) \frac{\partial \varphi}{\partial x_i} dx + \sum_{i,j=1}^3 \int_{\Omega} \bar{\lambda}_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_2} \frac{\tau_2^s}{s!} \frac{\partial^s \theta}{\partial t^s} \right) \frac{\partial \varphi}{\partial x_i} dx - \\ & - \Theta_0 \sum_{i,j=1}^3 \int_{\Omega} \eta_{ij} e_{ij} \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+2} \mathbf{u}}{\partial t^{r+2}} \right) \varphi dx = \int_{\Omega} f^\theta \varphi dx - \int_{\Gamma_1^\theta} g^\theta \varphi d\Gamma, \end{aligned} \quad (8)$$

where $f^\theta = \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \tilde{f}^\theta}{\partial t^{r+1}}$, $g^\theta = \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \tilde{g}^\theta}{\partial t^{r+1}}$. Therefore, if $\mathbf{u} = (u_i)_{i=1}^3$ and θ are solutions of equations (1), (2) and satisfy boundary conditions (3), (4), then $\mathbf{u} = (u_i)_{i=1}^3$ and θ are solutions of (7), (8). Conversely, if $\mathbf{u} = (u_i)_{i=1}^3$ and θ are smooth enough solutions of equations (7), (8), then by using the Green identity and density arguments, we obtain that $\mathbf{u} = (u_i)_{i=1}^3$ and θ satisfy equations (1), (2) and boundary conditions (3), (4). So, the initial-boundary value problem (1)-(5) corresponding to the nonclassical dynamic three-dimensional model of thermoelastic solids with three phase-lags is equivalent to equations (7), (8), together with initial conditions (5) in the spaces of smooth enough functions.

We identify the unknown vector-function $\mathbf{u}$ and the function θ with vector-functions defined on $[0, T]$ taking values in suitable function spaces defined on Ω . By applying the integral relations (7) and (8) in the case of $n_0 = n_1 + 1$, $n_2 \leq n_0 - 1$, we obtain the following variational formulation of the initial-boundary value problem (1)-(5) in the spaces of vector-valued distributions: Find the unknown vector-function

$\mathbf{u}, \mathbf{u}', \dots, \mathbf{u}^{(n_0+1)} \in C([0, T]; \mathbf{V}(\Omega))$, $\mathbf{u}^{(n_0+2)} \in L^\infty(0, T; \mathbf{V}(\Omega))$, $\mathbf{u}^{(n_0+3)} \in L^\infty(0, T; \mathbf{L}^2(\Omega))$, $\theta, \theta', \dots, \theta^{(n_0)} \in C([0, T]; V^\theta(\Omega))$, $\theta^{(n_0+1)} \in L^\infty(0, T; V^\theta(\Omega))$, $\theta^{(n_0+2)} \in L^\infty(0, T; L^2(\Omega))$, which satisfy the following equations in the sense of distributions on $(0, T)$,

$$(\rho \mathbf{u}'', \mathbf{v})_{\mathbf{L}^2(\Omega)} + a(\mathbf{u}, \mathbf{v}) + b(\theta, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2(\Omega)} + (\mathbf{g}, \text{tr}_{\Gamma_1}(\mathbf{v}))_{\mathbf{L}^2(\Gamma_1)}, \quad \forall \mathbf{v} \in \mathbf{V}(\Omega), \quad (9)$$

$$\begin{aligned} & \left(\chi \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \theta^{(r+2)}, \varphi \right)_{L^2(\Omega)} + a_1^\theta \left(\sum_{s=0}^{n_1} \frac{\tau_1^s}{s!} \theta^{(s+1)}, \varphi \right) + a_2^\theta \left(\sum_{s=0}^{n_2} \frac{\tau_2^s}{s!} \theta^{(s)}, \varphi \right) - \Theta_0 b^\theta \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \mathbf{u}^{(r+2)}, \varphi \right) = \\ & = (f^\theta, \varphi)_{L^2(\Omega)} - (g^\theta, \text{tr}_{\Gamma_1^\theta}(\varphi))_{L^2(\Gamma_1^\theta)}, \quad \forall \varphi \in V^\theta(\Omega), \end{aligned} \quad (10)$$

together with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{u}_1, \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta_1, \quad \dots, \quad \theta^{(n_0+1)}(0) = \theta_{n_0+1}, \quad (11)$$

where $\mathbf{V}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega); \text{tr}_{\Gamma}(\mathbf{v}) = \mathbf{0} \text{ on } \Gamma_0\}$, $V^\theta(\Omega) = \{\varphi \in H^1(\Omega); \text{tr}_{\Gamma}(\varphi) = 0 \text{ on } \Gamma_0^\theta\}$,

$$a(\tilde{\mathbf{v}}, \mathbf{v}) = \int_{\Omega} \sum_{i,j,p,q=1}^3 \mu_{ijpq} e_{pq}(\tilde{\mathbf{v}}) e_{ij}(\mathbf{v}) dx, \quad \forall \mathbf{v}, \tilde{\mathbf{v}} \in \mathbf{H}^1(\Omega),$$

$$b(\varphi, \mathbf{v}) = b^\theta(\mathbf{v}, \varphi) = \int_{\Omega} \varphi \sum_{i,j=1}^3 \eta_{ij} e_{ij}(\mathbf{v}) dx, \quad \forall \varphi \in L^2(\Omega), \mathbf{v} \in \mathbf{H}^1(\Omega),$$

$$a_1^\theta(\tilde{\varphi}, \varphi) = \int_{\Omega} \sum_{i,j=1}^3 \lambda_{ij} \frac{\partial \tilde{\varphi}}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx, \quad a_2^\theta(\tilde{\varphi}, \varphi) = \int_{\Omega} \sum_{i,j=1}^3 \bar{\lambda}_{ij} \frac{\partial \tilde{\varphi}}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx, \quad \forall \varphi, \tilde{\varphi} \in H^1(\Omega),$$

$(\cdot, \cdot)_{\mathbf{L}^2(\Omega)}$, $(\cdot, \cdot)_{L^2(\Omega)}$, $(\cdot, \cdot)_{\mathbf{L}^2(\Gamma_1)}$ and $(\cdot, \cdot)_{L^2(\Gamma_1^\theta)}$ are scalar products in the spaces $\mathbf{L}^2(\Omega)$, $L^2(\Omega)$, $\mathbf{L}^2(\Gamma_1)$ and $L^2(\Gamma_1^\theta)$, respectively.

For problem (9)-(11), which is equivalent to the initial-boundary value problem (1)-(5) in the spaces of smooth enough functions, the following theorem is valid.

Theorem. Suppose that Ω is a bounded Lipschitz domain, $n_0 = n_1 + 1$, $n_2 \leq n_0 - 1$, $\rho \in C^{n_0+1}(\bar{\Omega})$, $\mu_{ijpq} \in C^{n_0+1,1}(\bar{\Omega})$, $\eta_{ij} \in C^{n_0+1,1}(\bar{\Omega})$, $\lambda_{ij}, \bar{\lambda}_{ij} \in C^{0,1}(\bar{\Omega})$, $\chi \in L^\infty(\Omega)$, $i, j, p, q = 1, 2, 3$, $\rho(x) > c_\rho = \text{const} > 0$, $\chi(x) > c_\chi = \text{const} > 0$, for almost all $x \in \Omega$, and the symmetry conditions (6) and the following positive definiteness conditions are valid

$$\sum_{i,j,p,q=1}^3 \mu_{ijpq}(x) \varepsilon_{pq} \varepsilon_{ij} \geq c_\mu \sum_{i,j=1}^3 (\varepsilon_{ij})^2, \quad \forall \varepsilon_{ij} \in \mathbf{R}, \quad \varepsilon_{ij} = \varepsilon_{ji}, \quad i, j = 1, 2, 3, \quad c_\mu = \text{const} > 0, \quad x \in \Omega,$$

$$\sum_{p,q=1}^3 \lambda_{pq}(x) \varepsilon_p \varepsilon_q \geq c_\lambda \sum_{p=1}^3 (\varepsilon_p)^2, \quad \sum_{p,q=1}^3 \bar{\lambda}_{pq} \varepsilon_p \varepsilon_q \geq c_{\bar{\lambda}} \sum_{p=1}^3 (\varepsilon_p)^2, \quad \forall \varepsilon_p \in \mathbf{R}, \quad p = 1, 2, 3, \quad c_\lambda, c_{\bar{\lambda}} = \text{const} > 0, \quad x \in \Omega,$$

and $\tau_0 > 0$, $\tau_1 > 0$. If $\mathbf{f}^{(k)} \in C([0, T]; \mathbf{H}^{n_0+1-k}(\Omega))$, $k = 0, \dots, n_0$, $\mathbf{f}^{(n_0+1)}, \mathbf{f}^{(n_0+2)} \in L^2(0, T; \mathbf{L}^2(\Omega))$, $\mathbf{g}^{(\bar{k})} \in L^2(0, T; \mathbf{L}^{4/3}(\Gamma_1))$, $\bar{k} = 0, \dots, n_0 + 3$, $f^\theta, f^{\theta'} \in L^2(0, T; L^2(\Omega))$, $g^\theta, g^{\theta'}, g^{\theta''} \in L^2(0, T; L^{4/3}(\Gamma_1^\theta))$, and the initial conditions $\mathbf{u}_0 \in \mathbf{H}^{n_0+3}(\Omega) \cap \mathbf{V}(\Omega)$, $\mathbf{u}_1 \in \mathbf{H}^{n_0+2}(\Omega) \cap \mathbf{V}(\Omega)$, $\theta_{\bar{k}} \in H^{n_0+2-\bar{k}}(\Omega) \cap V^\theta(\Omega)$, $\bar{k} = 0, \dots, n_0$, $\theta_{n_0+1} \in V^\theta(\Omega)$, satisfy the following compatibility conditions

$$\begin{aligned} g_i^{(k)}(0) &= \text{tr}_{\Gamma_1} \left(\sum_{j=1}^3 \left(\sum_{p,q=1}^3 \mu_{ijpq} e_{pq}(\mathbf{u}_k) + \eta_{ij} \theta_k \right) \nu_j \right), \quad i = 1, 2, 3, \quad k = 0, 1, 2, \dots, n_0 + 1, \\ g^\theta(0) &= -\text{tr}_{\Gamma_1^\theta} \left(\sum_{i,j=1}^3 \lambda_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_1} \frac{\tau_1^s}{s!} \theta_{s+1} \right) \nu_i \right) - \text{tr}_{\Gamma_1^\theta} \left(\sum_{i,j=1}^3 \bar{\lambda}_{ij} \frac{\partial}{\partial x_j} \left(\sum_{s=0}^{n_2} \frac{\tau_2^s}{s!} \theta_s \right) \nu_i \right), \end{aligned}$$

where $\mathbf{v} = (\nu_i)_{i=1}^3$ is the unit outward normal vector to Γ ,

$$u_{ki} = \frac{1}{\rho} \left(\sum_{j=1}^3 \frac{\partial}{\partial x_j} \left(\sum_{p,q=1}^3 \mu_{ijpq} e_{pq}(\mathbf{u}_{k-2}) + \eta_{ij} \theta_{k-2} \right) + f_i^{(k-2)}(0) \right), \quad i = 1, 2, 3, \quad k = 2, 3, \dots, n_0 + 1,$$

then the initial-boundary value problem (9)-(11) possesses a unique solution, which continuously depends on the given data, i.e. the mapping

$$(\mathbf{u}_0, \mathbf{u}_1, (\theta_k)_{k=0}^{n_0+1}, (\mathbf{f}^{(\bar{k})})_{\bar{k}=0}^{n_0}, \mathbf{f}^{(n_0+1)}, (\mathbf{g}^{(\bar{k})})_{\bar{k}=0}^{n_0+2}, f^\theta, g^\theta, g^{\theta'}) \rightarrow ((\mathbf{u}^{(k)})_{k=0}^{n_0+1}, \mathbf{u}^{(n_0+2)}, (\theta^{(\bar{k})})_{\bar{k}=0}^{n_0}, \theta^{(n_0+1)})$$

is linear and continuous from space

$$\mathbf{H}^{n_0+2}(\Omega) \times \mathbf{H}^{n_0+1}(\Omega) \times \prod_{k=0}^{n_0+1} H^{n_0+1-k}(\Omega) \times \prod_{\bar{k}=0}^{n_0} C([0, T]; \mathbf{H}^{n_0-\bar{k}}(\Omega)) \times L^2(0, T; \mathbf{L}^2(\Omega)) \times \\ \times (L^2(0, T; \mathbf{L}^{4/3}(\Gamma_1)))^{n_0+3} \times L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^{4/3}(\Gamma_1^\theta)) \times L^2(0, T; L^{4/3}(\Gamma_1^\theta))$$

to space

$$(C([0, T]; \mathbf{V}(\Omega)))^{n_0+2} \times C([0, T]; \mathbf{L}^2(\Omega)) \times (C([0, T]; V^\theta(\Omega)))^{n_0+1} \times C([0, T]; L^2(\Omega)),$$

and the following energy equality is valid:

$$E(t) = E(0) + L(\mathbf{u}, \theta, \mathbf{f}, f^\theta, \mathbf{g}, g^\theta)(t), \quad \forall t \in [0, T],$$

where

$$E(t) = \frac{1}{2} \left(\rho \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+2} \mathbf{u}(t)}{\partial t^{r+2}}, \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+2} \mathbf{u}(t)}{\partial t^{r+2}} \right)_{\mathbf{L}^2(\Omega)} + \frac{1}{2} a \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \mathbf{u}(t)}{\partial t^{r+1}}, \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \mathbf{u}(t)}{\partial t^{r+1}} \right) + \\ + \frac{1}{2\Theta_0} \left(\chi \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \theta(t)}{\partial t^{r+1}}, \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \theta(t)}{\partial t^{r+1}} \right)_{L^2(\Omega)} + \frac{1}{\Theta_0} \int_0^t \sum_{s=0}^{n_0-1} \frac{(\tau_0 \tau_1)^s}{(s!)^2} a_1^\theta(\theta^{(s+1)}(\tau), \theta^{(s+1)}(\tau)) d\tau + \\ + \frac{1}{2\Theta_0} \sum_{s=0}^{n_0-1} \frac{\tau_0^{s+1} \tau_1^s}{(s+1)!s!} a_1^\theta(\theta^{(s+1)}(t), \theta^{(s+1)}(t)) + \frac{1}{2\Theta_0} \sum_{s=1}^{n_0-1} \frac{\tau_0^{s-1} \tau_1^s}{(s-1)!s!} a_1^\theta(\theta^{(s)}(t), \theta^{(s)}(t)) + \\ + \frac{1}{\Theta_0} \int_0^t \sum_{r=0}^{n_0-1} \sum_{s=0}^{n_0-1} \frac{\tau_0^r \tau_1^s}{r!s!} a_1^\theta(\theta^{(s+1)}(\tau), \theta^{(r+1)}(\tau)) d\tau + \frac{1}{\Theta_0} \sum_{s=0}^{n_0-2} \frac{\tau_0^{n_0} \tau_1^s}{n_0!s!} a_1^\theta(\theta^{(s+1)}(t), \theta^{(n_0)}(t)) - \\ - \frac{1}{\Theta_0} \int_0^t \sum_{s=0}^{n_0-2} \frac{\tau_0^{n_0} \tau_1^s}{n_0!s!} a_1^\theta(\theta^{(s+2)}(\tau), \theta^{(n_0)}(\tau)) d\tau + \frac{1}{\Theta_0} a_2 \left(\sum_{r=0}^{n_2} \frac{\tau_2^r}{r!} \frac{\partial^r \theta(t)}{\partial t^r}, \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^r \theta(t)}{\partial t^r} \right) - \\ - \frac{1}{\Theta_0} \int_0^t a_2^\theta \left(\sum_{r=0}^{n_2} \frac{\tau_2^r}{r!} \frac{\partial^{r+1} \theta(\tau)}{\partial t^{r+1}}, \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^r \theta(\tau)}{\partial t^r} \right) d\tau, \\ L(\mathbf{u}, \theta, \mathbf{f}, f^\theta, \mathbf{g}, g^\theta)(t) = \int_0^t \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \mathbf{f}(\tau)}{\partial t^{r+1}}, \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+2} \mathbf{u}(\tau)}{\partial t^{r+2}} \right)_{\mathbf{L}^2(\Omega)} d\tau + \\ + \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \mathbf{g}(t)}{\partial t^{r+1}}, \mathbf{tr}_{\Gamma_1} \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \mathbf{u}(t)}{\partial t^{r+1}} \right) \right)_{\mathbf{L}^2(\Gamma_1)} - \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \mathbf{g}(0)}{\partial t^{r+1}}, \mathbf{tr}_{\Gamma_1} \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \mathbf{u}(0)}{\partial t^{r+1}} \right) \right)_{\mathbf{L}^2(\Gamma_1)} - \\ - \int_0^t \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+2} \mathbf{g}(\tau)}{\partial t^{r+2}}, \mathbf{tr}_{\Gamma_1} \left(\sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \mathbf{u}(\tau)}{\partial t^{r+1}} \right) \right)_{\mathbf{L}^2(\Gamma_1)} d\tau + \frac{1}{\Theta_0} \int_0^t \left(f^\theta(\tau), \sum_{r=0}^{n_0} \frac{\tau_0^r}{r!} \frac{\partial^{r+1} \theta(\tau)}{\partial t^{r+1}} \right)_{L^2(\Omega)} d\tau - \\ - \frac{1}{\Theta_0} \int_0^t \left(g^\theta(\tau), \mathbf{tr}_{\Gamma_1^\theta} \left(\sum_{r=0}^{n_0-1} \frac{\tau_0^r}{r!} \theta^{(r+1)}(\tau) \right) \right)_{L^2(\Gamma_1^\theta)} d\tau - \frac{1}{\Theta_0} \left(g^\theta(t), \mathbf{tr}_{\Gamma_1^\theta} \left(\frac{\tau_0^{n_0}}{n_0!} \theta^{(n_0)}(t) \right) \right)_{L^2(\Gamma_1^\theta)} + \\ + \frac{1}{\Theta_0} \left(g^\theta(0), \mathbf{tr}_{\Gamma_1^\theta} \left(\frac{\tau_0^{n_0}}{n_0!} \theta^{(n_0)}(0) \right) \right)_{L^2(\Gamma_1^\theta)} + \frac{1}{\Theta_0} \int_0^t \left(g^{\theta'}(\tau), \mathbf{tr}_{\Gamma_1^\theta} \left(\frac{\tau_0^{n_0}}{n_0!} \theta^{(n_0)}(\tau) \right) \right)_{L^2(\Gamma_1^\theta)} d\tau.$$

მათემატიკა

თერმოდრეკადი სხეულების სამფაზიანი დაგვიანებით დროით მაღალი რიგის დიფერენციალური მოდელების გამოკვლევის შესახებ

გია ავალიშვილი*, მარიამ ავალიშვილი**

* ივანე ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი, ზუსტ და საბუნებისმეტყველო მეცნიერებათა ფაკულტეტი, საქართველო

** საქართველოს უნივერსიტეტი, მეცნიერებისა და ტექნოლოგიების სკოლა, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ელიზბარ ნადარაიას მიერ)

ნაშრომში განხილულია თერმოდრეკადი სხეულების სამფაზიანი დაგვიანებით დინამიკური სამგანზომილებიანი მოდელის დროითი ცვლადის მიმართ მაღალი რიგის დიფერენციალური აპროქსიმაცია. გამოკვლეულია სამი ფაზის წანაცვლების პარამეტრზე დამოკიდებული არაკლასიკური მოდელის შესაბამისი საწყის-სასაზღვრო ამოცანა შერეული სასაზღვრო პირობებით, სადაც საზღვრის გარკვეულ ნაწილებზე გადაადგილება და ტემპერატურა ნულის ტოლია, ხოლო საზღვრის შესაბამის დარჩენილ ნაწილებზე მოცემულია ზედაპირული ძალის და გარე ნორმალის გასწვრივ სითბოს ნაკადის სიმკვრივეები. მიღებულია საწყის-სასაზღვრო ამოცანის ვარიაციული ფორმულირება და სათანადო ვექტორული მნიშვნელობების მქონე განაწილებების სივრცეებში დამტკიცებულია არსებობის და ერთადერთობის შედეგი და ამონახსნის უწყვეტად დამოკიდებულება მონაცემებზე.

REFERENCES

- Avalishvili, G., & Avalishvili, M. (2014). On approximation of Lord-Shulman model for thermoelastic plates with variable thickness by two-dimensional problems. *Bull. Georg. Natl. Acad. Sci.*, 8(2), 4-14.
- Avalishvili, G., & Avalishvili, M. (2023). On investigation of nonclassical three-dimensional model for thermoelastic solids with three phase-lags. *Bull. Georg. Natl. Acad. Sci.*, 17(3), 7-14.
- Avalishvili, G., Avalishvili, M., & Müller, W.H. (2024). On the approximation of a nonclassical three-dimensional model with three phase-lags for thermoelastic plates by two-dimensional problems. *Bull. Georg. Natl. Acad. Sci.*, 18(3), 7-17.
- Avalishvili, G., & Avalishvili, M. (2025). On high-order time differential dual-phase-lag models of thermoelastic solids. *Bull. Georg. Natl. Acad. Sci.*, 19(3), 13-20.
- Borgmeyer, K., Quintanilla, R., & Racke, R. (2014). Phase-lag heat conduction: decay rates for limit problems and well-posedness. *Journal of Evolution Equations*, 14, 863–884. <https://doi.org/10.1007/s00028-014-0242-6>
- Chiriță, S. (2018) High-order approximations of three-phase-lag heat conduction model: some qualitative results, *Journal of Thermal Stresses*, 41(5), 608-626. <https://doi.org/10.1080/01495739.2017.1397494>
- Chiriță, S., D'Apice, C., & Zampoli, V. (2016). The time differential three-phase-lag heat conduction model: Thermodynamic compatibility and continuous dependence. *International Journal of Heat and Mass Transfer*, 102, 226-232. <http://dx.doi.org/10.1016/j.ijheatmasstransfer.2016.06.019>
- Chiriță, S., & Ciarletta, M. (2025). Structural stability of a three-phase-lag heat conduction model for a rigid conductor. *Journal of Thermal Stresses*, 48(12), 1584-1603. <https://doi.org/10.1080/01495739.2025.2477560>
- Ciarlet, P.G. (2013). *Linear and nonlinear functional analysis with applications: with 401 problems and 52 figures*. Society for Industrial and Applied Mathematics.
- Dautray, R., & Lions, J.-L. (2000). *Mathematical analysis and numerical methods for science and technology, Volume 5: Evolution problems I*. Springer-Verlag.
- Dreher, M., Quintanilla, R., & Racke, R. (2009). Ill-posed problems in thermomechanics. *Applied Mathematics Letters*, 22(9), 1374-1379. <https://doi.org/10.1016/j.aml.2009.03.010>
- Green, A.E., & Naghdi, P.M. (1991). A re-examination of the basic postulates of thermomechanics. *Proceedings of the Royal Society A*, 432(1885), 171-194. <https://doi.org/10.1098/rspa.1991.0012>
- Kumar, R., & Mukhopadhyay, S. (2010) Analysis of the effects of phase-lags on propagation of harmonic plane waves in thermoelastic media. *Computational Methods in Science and Technology*, 16(1), 19-28. <https://doi.org/10.12921/cmst.2010.16.01.19-28>
- Lord, H.W., & Shulman, Y. (1967). A generalized dynamical theory of thermoelasticity. *Journal of the Mechanics and Physics of Solids*, 15(5), 299-309. [https://doi.org/10.1016/0022-5096\(67\)90024-5](https://doi.org/10.1016/0022-5096(67)90024-5)
- McLean, W. (2000). *Strongly elliptic systems and boundary integral equations*. Cambridge University Press.
- Quintanilla, R., & Racke, R. (2008). A note on stability in three-phase-lag heat conduction. *International Journal of Heat and Mass Transfer*, 51(1-2), 24-29. <https://doi.org/10.1016/j.ijheatmasstransfer.2007.04.045>
- Roy Choudhuri, S.K. (2007). On a thermoelastic three-phase-lag model. *Journal of Thermal Stresses*, 30(3), 231-238. <https://doi.org/10.1080/01495730601130919>
- Shakeriaski, F., Ghodrat, M., Escobedo-Diaz, J., & Behnia, M. (2021). Recent advances in generalized thermoelasticity theory and the modified models: a review. *Journal of Computational Design and Engineering*, 8(1), 15-35. <https://doi.org/10.1093/jcde/qwaa082>
- Tzou, D.Y. (1995). A unified approach for heat conduction from macro to micro-scales. *Journal of Heat Transfer*, 117(1), 8-16. <https://doi.org/10.1115/1.2822329>
- Zampoli, V., & Landi, A. (2017) A domain of influence result about the time differential three-phase-lag thermoelastic model, *Journal of Thermal Stresses*, 40(1), 108-120. <http://dx.doi.org/10.1080/01495739.2016.1195242>

Received June, 2026