

Generalization of Landau's Theorem for the Number of Discrete Levels in Energy-Dependent Potentials

Anzor Khelashvili*, Teimuraz Nadareishvili*,**

* Academy Member, Institute of High Energy Physics, Ivane Javakhishvili Tbilisi State University, Georgia

** Department of Physics, Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University, Georgia

Abstract. This paper studies a modification of the classical Landau theorem in cases where the interaction potential is energy-dependent and exhibits quadratic decay at infinity. It is shown that the theorem generalizes and retains its original content. An explicit example of an energy-dependent quasipotential is constructed to confirm the validity of the generalized theorem. © 2026 Bull. Georg. Natl. Acad. Sci.

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Introduction

In quantum mechanics textbooks (Landau and Lifshitz, 1989), a proof is provided regarding the number of discrete energy levels for potentials that decrease inversely with the square of the distance at infinity:

$$V(R) \approx -\frac{\beta}{r^2}, \quad (\beta > 0). \quad (1)$$

At the same time, any of their forms are allowed at short distances. The following parameter plays a critical role

$$\gamma = \frac{2m\beta}{\hbar^2} - l(l+1). \quad (2)$$

It has been shown that, if $\gamma < 1/4$, the zeros of the sought radial function can be located only at finite distances from the origin and, therefore, the number of levels, in any case, will be finite.

Conversely, if $\gamma < 1/4$, an infinite number of negative-energy levels is expected. To obtain this result, the upper threshold of the discrete spectrum $E = 0$ is substituted into the reduced Schrödinger equation, and the asymptotic solutions of the equation are considered, when $r \gg 1$. It is essential that the parameter β defining the potential is a constant number.

In atomic and nuclear physics, when relativistic effects are taken into account, the interaction potential often depends not only on interparticle distance, but also on the total energy of the system E . In such cases, the asymptotic behavior of the potential energy may still obey the r^{-2} law, but its strength depends on the energy $\beta = \beta(E)$, i.e. $V_0 = V_0(E)$. It is clear that Landau's assertion is no longer valid, since substituting

the value $E = 0$ may not be permissible to generalize the theorem. To address this problem, consider the reduced Schrödinger equation in its asymptotic form

$$\chi''(r) - \left(k^2 - \frac{\gamma(E)}{r^2} \right) \chi(r) = 0, \quad (3)$$

where $k = \sqrt{\frac{2m|E|}{\hbar^2}}$ and $E < 0$ (for the bound states). The general solution of this equation is the superposition of Bessel functions with imaginary arguments (Chilashvili, 1973):

$$\chi(r) = c_1 \sqrt{r} J_{\sqrt{1/4 - \gamma(E)}}(ikr) + c_2 \sqrt{r} J_{-\sqrt{1/4 - \gamma(E)}}(ikr). \quad (4)$$

Since we are interested in bound states, we must choose the coefficients $c_{1,2}$ to obtain wave function decay at infinity, so that it will be normalizable. Among the Bessel functions, only the modified only so-called McDonald function exhibits exponentially decreasing asymptotics. To obtain this, we must assume that

$$c_1 = -c_2 = -\frac{\pi}{2 \sin(\pi v)}, \quad v = \sqrt{\frac{1}{4} - \gamma(E)}. \quad (5)$$

Then we obtain

$$\chi(r) \sim \sqrt{r} K_{\sqrt{1/4 - \gamma(E)}}(kr). \quad (6)$$

For the following analysis, we make use of the properties of the McDonald function. It is well known that the McDonald function has real zeros only for purely imaginary indices. Therefore, we conclude that:

(a) If $\gamma(E) < \frac{1}{4}$, the radial function $\chi(r)$ has no zeros (nodes) at infinity, i.e. the Schrödinger equation either has no discrete energy levels at all, or only a finite number of them; (b) If $\gamma(E) > 1/4$, the index of the McDonald function is purely imaginary and the $\chi(r)$ function will have nodes at infinity. Therefore, it is expected that the Schrödinger equation is expected to have an infinite number of discrete energy levels.

Thus, we have obtained a generalization of the above theorem for energy-dependent potentials. It is clear that the assertion applies to energy levels lying between a given value and a critical value $E(E_{cr})$, which is determined by the condition $\gamma(E_{cr}) = \frac{1}{4}$. In conclusion, let us emphasize that numerous examples of energy-dependent potentials arise for which the above generalization of Landau's theorem is applicable.

Specific Example

In the hydrogen atom problem, when taking into account the relativistic effects, R. Faustov obtained the following type of interaction quasipotential (Faustov, 1968) in momentum space:

$$V(M; p, q) = \frac{(2me)^2}{|p - q|(\varepsilon - |p - q| + i0)}, \quad (8)$$

where $\varepsilon = M - 2m$ is the binding energy. In this case, we have an integral equation of the Fredholm type. When $\varepsilon < 0$, we can neglect ε in the denominator and then (7) will transform into an ordinary Coulomb potential, and the standard methods of the Fredholm theory can be applied. Conversely, when retaining a non-zero ε , the quasipotential equation in coordinate space transforms into a Schrödinger type equation (Khelashvili and Kiknadze, 1997):

$$\left(\varepsilon + \frac{\nabla^2}{m}\right)\Psi_\varepsilon(r) = V_\varepsilon(r)\Psi_\varepsilon(r), \quad (9)$$

where the potential is

$$V_\varepsilon(r) = -\frac{2\alpha}{\pi} \frac{f(\varepsilon r)}{r}, \quad (10)$$

while $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant and $\varepsilon = -E > 0$. In this expression, the following function $f(\varepsilon r)$ is denoted by Bateman and Erdi (1963)

$$f(\varepsilon r) = Ci(\varepsilon r)\sin(\varepsilon r) - si(\varepsilon r)\cos(\varepsilon r), \quad (11)$$

while the functions included here are the integral cosine and sine, respectively:

$$Ci(x) = \int_0^x \frac{\cos t}{t} dt, \quad si(x) = \int_x^\infty \frac{\sin t}{t} dt. \quad (12)$$

Despite the trigonometric functions, it has a rather smooth behavior (see Figure).

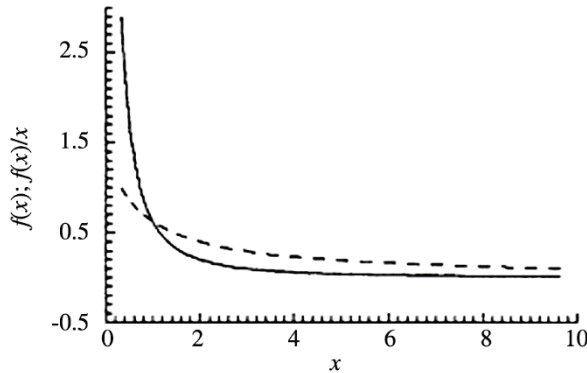


Figure. Functions: $f(x)$ (dashed line) and $f(x)/x$ (solid line).

Namely, it is positive definite function and

$$f(x) \approx \frac{\pi}{2} + x \ln x + (\gamma - 1)x - \frac{\pi}{4}x^2 + O(x^3), \quad x \ll 1 \quad (13)$$

$$f(x) \approx \frac{1}{x} \left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \dots \right), \quad x \gg 1, \quad (14)$$

where $\gamma = 0.5772$ is Euler's constant.

Taking into account this formula, we conclude that when $\varepsilon > 0$, the potential energy (10) has the following behavior (5):

$$V_\varepsilon(r) \approx -\frac{\alpha}{r} \left[1 + \frac{2}{\pi} \varepsilon r \ln(\varepsilon r) + \frac{2}{\pi} (\gamma - 1) \varepsilon r + O((\varepsilon r)^2) \right]; \quad \varepsilon r \ll 1. \quad (15)$$

$$V_{\varepsilon r}(r) \approx -\frac{2\alpha}{\pi \varepsilon r^2}, \quad \varepsilon r \gg 1. \quad (16)$$

At the same time, if we strictly take $\varepsilon = 0$ in the potential energy, then according to (8) we obtain the Coulomb potential (Kapshai et al. 1988)

$$V_{\varepsilon=0}(r) \approx -\frac{\alpha}{r}. \quad (17)$$

Evidently, setting the binding energy in the potential to zero, essentially alters its asymptotic behavior. For small but non-zero binding energies, the asymptotic regime of the potential differs fundamentally from that of Coulomb potential. According to expression (15), the potential then acquires a form to which the theorem above proved can be applied. In this case, we choose:

$$\gamma(E) = \frac{4m\alpha}{\pi\varepsilon} - l(l+1). \quad (18)$$

Then the solution to the Schrödinger equation will have the following form

$$\chi(r) \sim \sqrt{r} K_{\sqrt{\frac{1}{4} - \gamma(E)}}(kr). \quad (19)$$

Due to the above properties of the nodes of the Macdonald function, we conclude that: a) if there are no discrete levels, or their number is finite then $\gamma(E) < 1/4$. b) if $\gamma(E) > 1/4$ then an infinite number of levels is expected.

It is interesting to find out whether the expected new solutions have any effect on the spectrum of the hydrogen atom. For simplicity, we consider only the S -states, where $l = 0$ and then

$$\gamma_s(E) = \beta(E) = \frac{4m\alpha}{\pi\varepsilon}. \quad (20)$$

First of all, we note that in the region of the ordinary Balmer spectrum, where

$$\varepsilon = \frac{m\alpha}{2n^2}, \quad (21)$$

which is a rather large number $\beta > 137$ for the experimental value of the fine structure constant $\alpha \approx \frac{1}{137}$.

Therefore, the existence of the Balmer spectrum does not contradict the conclusion of the obtained theorem, although it concerns a potential different from the Coulomb one.

Along with the condition $\gamma(E) > 1/4$, we must require that

$$\sqrt{\gamma(E) - 1/4} > kr \quad (22)$$

or

$$\frac{2\alpha}{\pi} > \frac{\varepsilon}{2m} \left[\frac{1}{4} + l(l+1) \right] + (\varepsilon r)^2. \quad (23)$$

This occurs because the smallest positive root of the MacDonnald function with an imaginary index lies in the region where the modulus of the index exceeds that of the argument, which is indicated by the obvious asymptotic formulas found in mathematical publications.

Within this asymptotic regime, the condition defined by inequality (23) can only be satisfied for sufficiently large values of α . Therefore, the emergence of any new levels within the atomic nucleus is not expected.

ფიზიკა

ლანდაუს თეორემის განზოგადება დისკრეტული დონეების რაოდენობის შესახებ ენერგიაზე დამოკიდებული პოტენციალებისათვის

ანზორ ხელაშვილი*, თეიმურაზ ნადარეიშვილი**,

* აკადემიის წევრი, ივანე ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი, მაღალი
ენერგიების ფიზიკის ინსტიტუტი, საქართველო

** ივანე ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი, ზუსტ და საბუნებისმეტყველო
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REFERENCES

- Bateman, H. and Erdelyi, A. (1963). The higher transcendental functions, vol. 2. Bombay-New Delhi.
Chilashvili, G. (1973). *Ori da sami natsilakis kvanturi mekanika* [Two- and three-body quantum mechanics], Tbilisi
State University.
Kapshai, V., Savrin, V. and Skachkov, B. (1986). O zavisimosti kvazipotentsiala ot polnoi energii dvukhchastichnoi
sistemy [Dependence of the quasipotential on the total energy of a two-particle system], *Theoretical and
Mathematical Physics*, 69, 3, 400.
Khelashvili, A., Kiknadze, N. (1997). Bound states in continuously induced by relativity, *Phys. A.*, 55, 2557.
Landau, L., Lifshitz, E. (1989). *Kvantovaia mekhanika* [Quantum mechanics]. M. 'Nauka'.
Faustov, G. (1968). The proton structure and hyperfine splitting of hydrogen energy levels, *Nuclear Physics*, 75, 669.

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