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Mathematics

Solving Linear Partial Differential Equations by Moving Least Squares Method

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ABSTRACT. In this work we consider a method for solving linear partial differential equations, specially heat and waves equations that describe behavior of temperature distribution and wave propagation in one or multidimensional environments by moving least squares procedure. We present some illustrative examples and compare our proposed method with other methods to show the efficiency of this method. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *partial differential equations; moving least squares; approximation.*

1. Introduction

Historically, Partial Differential Equations (PDE) originated from the study of surfaces in geometry and for solving a wide variety of problems in mechanics.

It is well known that most of the phenomena that arise in mathematical physics and engineering fields can be described by PDE. In Physics for example, the heat flow and the wave propagation phenomena are well described by partial differential equations [1, 2]. In ecology, most population models are governed by partial differential equations [3, 4].

The Moving Least Squares (MLS) as approximation method has been introduced by Shepard [5], in the lowest order case and generalized to higher degree by Lancaster and Salkauskas [6].

In the recent years some works have been done on MLS method and it had been used in many fields of mathematics to reach an approximate solution of a problem at a fixed point, especially in the scattered data approximation [7-13]. Some works have been done for error estimate and error analyzes of the MLS method [14, 15]. Let Ω be a subset on $\mathbb{R}^d$ and $\partial\Omega$ be the bound of this set. In this work we consider the following equation

$$\begin{cases} \mathcal{A}(u)=f, & x\in\Omega\setminus\partial\Omega, \\ \mathcal{B}(u)=q, & x\in\partial\Omega. \end{cases} \tag{1.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are linear operators on an unknown function u, also the function f named *source term* and q are given. In this paper we want to use MLS method at a set of points such as $x_k\in\Omega$ and construct a set of approximations of the unknown function u over the domain Ω and finally we present the Root Mean Square (RMS) error.

There are various methods for solving a PDE such as Adomian Decomposition Method (ADM), and the Variational Iteration Method (VIM), Finite Integration Method (FIM), Finite Element Method (FEM) and Boundary Element Method (BEM), Radial Basis Functions (RBFs) and vary other methods [1, 4, 16, 17, 18]. In this paper we compare the proposed MLS method with the methods which W. Li et al proposed in [18].

The structure of this paper is as follows: in Section 2 we reintroduce the moving least squares method in the general form. In Section 3 we use MLS method for solving PDEs. There are some examples in Section 4 to demonstrate the efficiency and accuracy of the proposed method. Section 5 consists of a brief conclusion.

2. Moving Least Squares

Suppose that u be a multivariate function on a domain $\Omega \in \mathbb{R}^d$. We want to approximate u at a certain point x by using some points in a neighborhood of x. Let the values of u on a set of nodes $X = \{x_1, x_2, \ldots, x_N\} \subseteq \Omega$ from its domain are known. We consider an approximate value for u, on a given point x, as follows:

$$u(x) \simeq \hat{u}(x) = P^T(x)\boldsymbol{\alpha} = \sum_{j=1}^{m} \alpha_j p_j(x), \; x \in \Omega, \tag{2.1}$$

where $P(x_i) = (p1(x), p2(x), \cdots, pm(x))^T$ is a m-dimensional basis of functions and $\boldsymbol{\alpha}$ is a vector of parameters to be determined. Let Ω_x be a δ neighborhood of a fixed point $x \in \Omega$. The parameter δ, which usually called *smoothing length* or *dilatation parameter* in the mesh free literature, is a certain characteristic measure of the size of the support Ω_x. Thus for a fixed point x, we try to solve the following minimization problem

$$min\left\{ \sum_{i \in I_{x,\delta}} \left(u(x_i) - \sum_{j=1}^{m} \alpha_j p_j(x) \right)^2 w(x - x_i) : \alpha_j \in \mathbb{R} \;\;, j = 1, 2, \cdots, m \right\}, \tag{2.2}$$

where

$$I_{x,\delta} = \left\{ i \in \{1, 2, \ldots, N\} : \|x - x_i\|_2 \le \delta \right\},$$

and also $w : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative function with support in the unit ball $B(0,1)$, which is positive on the ball B (0, 0.5) and it is called weight function. Now for constructing the matrix form of problem (2.2) we introduce the following notations:

$$\begin{cases} \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T, \\ U_{\Omega_x} = \left(u(x_i) | i \in I_{x,\delta}\right)^T \in \mathbb{R}^{c_x}, \\ P_{\Omega_x} = \left(p_j(x_i)\right)_{i \in I_{x,\delta}\;, j \in \{1,2,\ldots,m\}} \in \mathbb{R}^{m \times c_x}, \\ W_{\Omega_x} = diag(w(x - x_i) | i \in I_{x,\delta}) \in \mathbb{R}^{c_x \times c_x}, \end{cases} \tag{2.3}$$

where c_x is the cardinal of $I_{x,\delta}$. By regarding to notations (2.3), problem (2.2) can be rewritten as follows:

$$min\{M(\alpha) | \alpha \in \mathbb{R}^m\}, \tag{2.4}$$

where M is a quadratic function with respect to m-tuple α, and has the following matrix form:

$$M(\alpha) = \sum_{i \in I_{x,\delta}} \left(\left(U_{\Omega_x}\right)_i - \left(P_{\Omega_x}{}^T \alpha\right)_i \right)^2 w_{\Omega_x}(x - x_i)$$

$$=\left(U_{\Omega_x}-P_{\Omega_x}{}^T\alpha\right)^T W_{\Omega_x}\left(U_{\Omega_x}-P_{\Omega_x}{}^T\alpha\right).$$

So that we have

$$M(\alpha)=U_{\Omega_x}W_{\Omega_x}U_{\Omega_x}-2\alpha^T P_{\Omega_x}W_{\Omega_x}U_{\Omega_x}+\alpha^T P_{\Omega_x}W_{\Omega_x}P_{\Omega_x}{}^T\alpha\,. \tag{2.5}$$

Suppose that with suitable conditions $\boldsymbol{\alpha}^*$ be solution of (2.4) it requires that first ingredient derivatives in $\boldsymbol{\alpha}^*$ must be zero, in other words the gradient vector should be zero at $\boldsymbol{\alpha}^*$:

$$\nabla M\left(\boldsymbol{\alpha}^*\right)=0,$$

thus

$$-2P_{\Omega_x}W_{\Omega_x}U_{\Omega_x}+2P_{\Omega_x}W_{\Omega_x}P_{\Omega_x}{}^T\alpha=0,$$

which implies that

$$\left[P_{\Omega_x}W_{\Omega_x}P_{\Omega_x}{}^T\right]\alpha=P_{\Omega_x}W_{\Omega_x}U_{\Omega_x}\,. \tag{2.6}$$

The system (2.6) is a system of m linear equations with m unknowns. If the number of points inside of m is less than the number of the basic functions Ω_x then the matrix of coefficients can be singular so that we have to apply a numerical method such as least squares to solve it, otherwise system (2.6) has a unique solution [19- 21]. We substitute the solution of the system (2.6) $\boldsymbol{\alpha}^*$ in (2.1) and achieve an approximation to function u in the neighborhood of x [13].

3. Approximating Partial Differential Equation's solution by MLS

We want to approximate inhomogeneous PDE's solution with boundary conditions by Moving Least Squares (MLS). For this purpose we consider a set of nodes over defined domain regard to the boundary conditions, and then we separate nodes into two groups, those are on bound and those are inside the domain.

For $X=\{x_1,x_2,\ldots,x_N\}\subseteq\Omega\subseteq\mathbb{R}^d$, we denote the set of boundary points by B and the set of inner points by I.

$$I^B_{x,\delta}=\left\{i\in I_{x,\delta}\ :x_i\in\partial\Omega\right\},$$

$$I^I_{x,\delta}=I_{x,\delta}\setminus I^B_{x,\delta}.$$

We use the space of multivariate polynomials of degree n with d variables for constructing the approximate solution of PDE (1.1). Let $\{p_1,p_2,\ldots,p_m\}$ be a basis of the space, so that $m=\binom{d+n}{n}$. We set $u(x)\simeq\sum_{j=1}^{m}\alpha_j p_j(x)$, as a solution of the Equation (1.1). We analogously set a fixed point $x\in\Omega$ and a δ neighborhood Ω_x of this point. In general Ω_x may includes some points of B and others in I, so we must solve the following minimization problem:

$$\begin{aligned} min\Bigg\{ & \sum_{k\in I_{x,\delta}^{B}} \left(u(x_k) - \sum_{j=1}^{m} \alpha_j p_j(x_k) \right)^2 w(x-x_k) \\ & + \sum_{l\in I_{x,\delta}^{I}} \left(u(x_l) - \sum_{j=1}^{m} \alpha_j p_j(x_l) \right)^2 w(x-x_l) \ : \alpha \in \mathbb{R}^m \Bigg\} \end{aligned} \tag{3.1}$$

where $w:\mathbb{R}^d \to \mathbb{R}$ is a weight function. Considering operators $\mathcal{A}$ and $\mathcal{B}$ in (1.1), (3.1) will be changed to the following form:

$$\begin{aligned} min\Bigg\{ & \sum_{k\in I_{x,\delta}^{B}} \left(q(x_k) - \mathcal{B}\left(\sum_{j=1}^{m} \alpha_j p_j(x_k) \right) \right)^2 w(x-x_k) \\ & + \sum_{l\in I_{x,\delta}^{I}} \left(\mathcal{A}\left(\sum_{j=1}^{m} \alpha_j p_j(x_l) \right) - f(x_l) \right)^2 w(x-x_l) : \alpha \in \mathbb{R}^m \Bigg\}. \end{aligned} \tag{3.2}$$

Considering $r = \sharp\, I_{x,\delta}^{B}$ and $s = \sharp\ I_{x,\delta}^{I}$ which notation $\sharp$ denotes the cardinal of a set, we construct the matrix form of the introduced problem in (3.2) by the following notations

$$\begin{cases} \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T , \\ Q = \left(q(x_k) \,|\, k \in I_{x,\delta}^{B}\right)^T \in \mathbb{R}^r , \\ P_B = \mathcal{B}\left(p_j(x_k)\right) \in \mathbb{R}^{m\times r}, \\ W_B = \mathrm{diag}\left(\mathrm{w}(x-x_k) \,|\, k \in I_{x,\delta}^{B}\right) \in \mathbb{R}^{r\times r}, \\ W_I = \mathrm{diag}\left(w(x-x_l) \,|\, l \in I_{x,\delta}^{I}\right) \in \mathbb{R}^{s\times s} , \\ F = \left(f(x_l) \,|\, l \in I_{x,\delta}^{I}\right)^T \in \mathbb{R}^s , \\ P_I = \left(\mathcal{A}\left(p_j(x_l)\right)\right) \in \mathbb{R}^{m\times s}. \end{cases} \tag{3.3}$$

Now we can rewrite the problem (3.2) as follows:

$$min\left\{ M(\alpha) \,|\, \alpha \in \mathbb{R}^m \right\}, \tag{3.4}$$

where M is a quadratic function of m-tuple $\boldsymbol{\alpha}$. $M(\boldsymbol{\alpha})$ has the following form:

$$M(\alpha) = \sum_{k\in I_{x,\delta}^{B}} \left(Q_k - \left(P_B^T \alpha\right)_k \right)^2 w(x-x_k) + \sum_{l\in I_{x,\delta}^{I}} \left(\left(P_I^T \alpha\right)^l - F_l \right)^2 w(x-x_l).$$

In regarding to notations (3.3) we have the following matrix form for $M(\boldsymbol{\alpha})$:

$$\begin{aligned} M(\alpha) = & \, Q^T W_B Q - 2\alpha^T P_B W_B Q + \alpha^T P_B W_B P_B^T \alpha \\ & + \alpha^T P_I W_I P_I^T \alpha - 2\alpha^T P_I W_I F + F^T W_I F. \end{aligned} \tag{3.5}$$

If the coefficient matrices are positive definite then $M(\boldsymbol{\alpha})$ has a minimum and we can use the necessary condition $\nabla M(\boldsymbol{\alpha}) = 0$ to find α^*. Thus

Table 1. RMS errors for Example 4.1

g , h \ n	1	2	3	4	5	6
g = 9, h = 3	1.27e-01	6.64e-02	3.18e-02	5.37e-02	2.76e-02	1.66e-02
g = 18, h = 6	1.41e-01	7.66e-02	3.46e-02	1.29e-02	2.68e-03	4.65e-04

$$\nabla M(\alpha) = -2P_B W_B Q + 2P_B W_B P_B^T \alpha + 2P_I W_I P_I^T \alpha - 2P_I W_I F = 0.$$

Finally we have the following system of linear equations

$$\left[P_B W_B P_B^T + P_I W_I P_I^T \right] \alpha = P_B W_B Q + P_I W_I F \,, \tag{3.6}$$

which is a linear system of m equations with m unknowns. We obtain α^* by solving the system (3.6) and by substituting α^* in $\sum_{j=1}^{m} \alpha_j p_j(x)$, we will have an approximation for $u(x)$.

4. Numerical Examples

In this section we approximate solution of inhomogeneous and homogeneous PDEs with boundary and initial conditions in some determined nodes from [4] and [18]. The Root Mean Square (RMS) errors for our method is demonstrating the accuracy of the method. RMS error is defined as follows: Let for $i = 1,2,\ldots,N_s$, $u(x_i)$ are exact values of solution function u at nodes x_i and $u_{appr}(x_i)$ are corresponding approximation values on those nodes, the RMS error is defined by

$$\text{RMS} = \sqrt{\frac{\sum_{i=1}^{N_s} \| u(x_i) - u_{appr}(x_i) \|^2}{N_s}} \tag{4.1}$$

[22].

Also in Examples 4.5 and 4.6, to compare the mentioned method with other methods we use other criteria such as average relative error ARE and average error AE which are defined in the following definition. With analogous assumptions in Definition of RMS, the average relative error ARE and the average error AE are

$$ARE = \frac{1}{N_s} \sum_{i=1}^{N_s} \frac{\| u(x_i) - u_{appr}(x_i) \|}{\max u(x)}, \tag{4.2}$$

and

$$AE = \frac{1}{N_s} \sum_{i=1}^{N_s} \| u(x_i) - u_{appr}(x_i) \|, \tag{4.3}$$

[18].

Table 2. RMS errors for Example 4.2

g , h \ n	1	2	3	4	5	6
g = 9, h = 3	7.53 e-02	3.07 e-02	1.79 e-02	6.47 e-02	9.53 e-03	2.28 e-03
g = 18,h =6	9.04 e-02	3.74 e-02	1.84 e-02	1.11 e-02	2.37 e-03	5.56 e-04

Table 3. RMS errors for Example 4.3

g , h \ n	1	2	3	4	5	6
g = 9, h = 3	2.70 e-01	8.67 e-02	1.32 e-02	3.47 e-03	4.49 e-04	1.72 e-04
g = 18, h = 6	3.24 e-01	7.94 e-02	1.15 e-02	2.68 e-03	3.72 e-04	7.72 e-05

In the following examples our goal is approximating the values of the unknown function u at some points in the region Ω. In Examples 4.1- 4.3, without loss of generality just for simplicity and in regarding to the initial and boundary conditions of problem, we take a regular nodes in Ω which their abscissas and ordinates are equally spaced and we call Δ that rectangular region. In other examples Ω may be different by regarding to conditions of problem.

Example 4.1. *Consider the following inhomogeneous heat equation*

$$u_t = u_{xx} + \sin x, \quad 0 < x < \pi,\ t \ge 0, \tag{4.4}$$

with boundary and initial conditions:

$$\text{BC} \ : \ u(0,t) = e^{-t},\ t \ge 0,$$

$$u(\pi,t) = -e^{-t},\ t \ge 0,$$

$$IC: \ u(x,0) = \cos x .$$

The exact solution of (4.4) with respect to its boundary and initial conditions, is

$$u(x,t) = \left(1-e^{-t}\right)\sin x + e^{-t}\cos x .$$

In order to approximate values of u we consider $(g+1)(h+1)$ regular nodes (their abscissas and ordinates are equally spaced) on $\Omega = [0,\pi]\times[0,T]$, $T=1$, that means the number of partitions in x and t directions are g and h, respectively and apply the proposed MLS method on those nodes. The weight function is $w(x,x_i) = \dfrac{1}{1+\|x-x_i\|_2^2}$. We use bivariate polynomials $(d=2)$ of degree at most n. RMS error for different amounts of g,h and different basis functions of degree n are shown in Table 1.

Example 4.2. *Consider the following homogeneous heat equation*

$$u_t = u_{xx} - u, \quad 0 < x < \pi,\ t \ge 0, \tag{4.5}$$

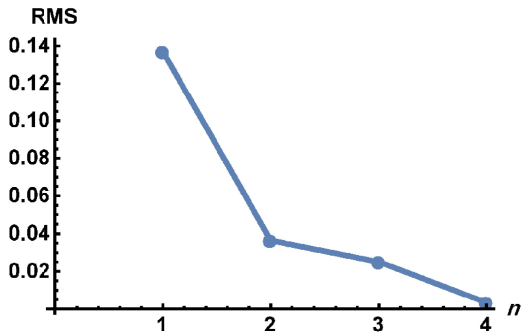
with boundary and initial conditions:

$$\text{BC} \ : \ u(0,t) = 0,\ t \ge 0,$$

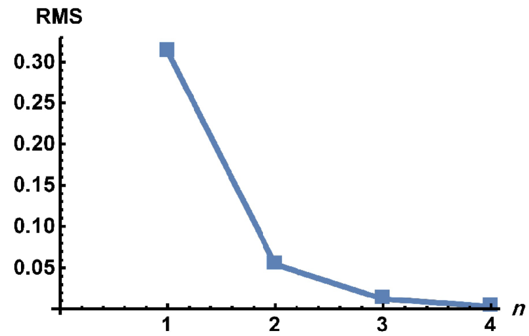
$$u(\pi,t) = 0,\ t \ge 0,$$

Table 4. RMS errors for Example 4.4

g , h, k \ n	1	2	3	4
g = 9, h = 3, k=2	1.37 e-01	3.65 e-02	2.51 e-02	3.47 e-03
g = 18, h = 6, k=4	3.13 e-01	5.04 e-02	1.22 e-02	2.81 e-03



(a) $g=2, h=3, k=2$



(b) $g=4, h=6, k=4$

Fig. 1. RMS error for various number of n and density of mesh for Example 4.4

$$IC: \quad u(x,0)=\sin x .$$

The exact solution of (4.5) regarding to above boundary and initial conditions, is

$$u(x,t)=e^{-2t}\sin x .$$

We set analogues assumptions such as those in the Example 4.1. RMS error for different amounts of *g, h* and various number of basic functions are shown in Table 2.

Example 4.3. *Consider the following inhomogeneous wave equation*

$$u_{tt}=u_{xx}-2,\ 0<x<\pi,\ t\geq 0, \tag{4.6}$$

with boundary and initial conditions:

$$BC: \quad u(0,t)=0, \qquad t\geq 0,$$

$$u(\pi,t)=\pi^2, \qquad t\geq 0,$$

$$IC: \quad u(x,0)=x^2,$$

$$u_t(x,0)=\sin x.$$

The exact solution of (4.6) under the above conditions, is

$$u(x,t)=x^2+\sin x \sin t.$$

By taking the integral of (4.6) two times and using the initial conditions, we have

Table 5. ARE for problem in Example 4.5 that are compared for various methods

method / N_s	MLS	OLA	MQ	LF	TPS
10	1.4823 e-11	1.9110 e-02	1.3707 e-02	1.7554 e-02	1.2985 e-02
20	1.5024 e-14	5.6890e-03	5.3650e-03	6.4620 e-03	4.4960 e-03
30	1.2051 e-14	1.3820 e-02	3.0830 e-03	3.671 e-03	2.3730 e-03

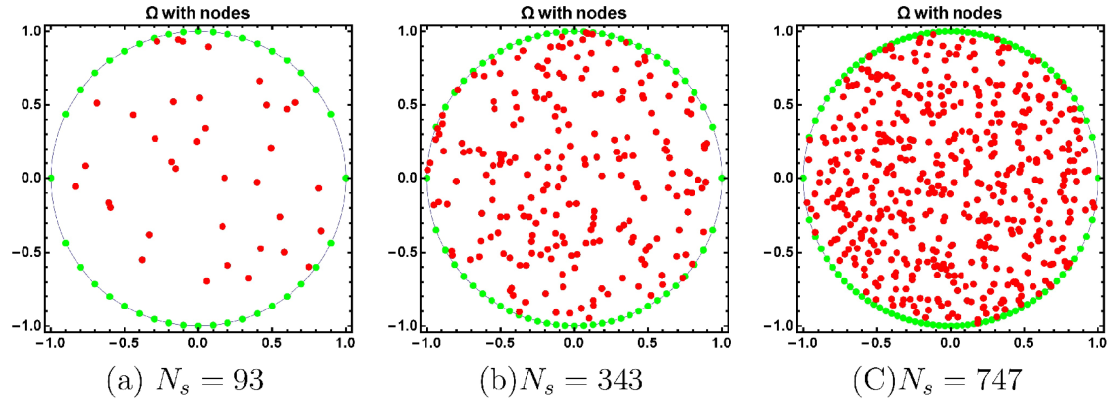


Fig. 2. The irregular distanced points in Ù for various Ns which are used in MLS method for Example 4.6, boundary points (green points) and inner points (red points).

$$u(x,t) = x^2 + \sin x - t^2 + \int_0^t \int_0^t u_{xx}(x,\xi)\, d\xi d\xi. \tag{4.7}$$

Therefore we try to solve the Equation (4.7) by MLS method so our operator is $\mathcal{A}(u) = u - \int_0^t \int_0^t u_{xx}\, d\xi d\xi$, with the source term $x^2 + \sin x - t^2$. In this case we can use the whole initial conditions. By using the same assumptions of Example 4.1 with $\delta = 2$, RMS errors are shown in Table 3.

Example 4.4. *Consider the following heat equation in two dimensional space,*

$$u_t = u_{xx} + u_{yy}, \qquad 0 < x, y < \pi, \qquad t \geq 0, \tag{4.8}$$

with boundary and initial conditions:

$$BC: \quad u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,0,t) = u(x,\pi,t) = 0,$$

$$IC: \quad u(x,y,0) = \sin x \sin y.$$

The exact solution of (4.8) respect to the given conditions, is

$$u(x,y,t) = e^{-2t} \sin x \sin y.$$

Using the same assumptions of previous Examples with this little difference that in this case we have three independent variables thus we use polynomials with three variable as basis also $\Omega \in \mathbb{R}^3$ and have three directions *x, y* and *t* which respectively *g, h* and *k* are the number of partitions. RMS errors which are shown in Table 4 and Figure 1, demonstrate the accuracy of the MLS method.

Example 4.5. *In this example we consider the following partial differential equation which has been considered in [18]*

Table 6. AE for problem in Example 4.6 which solved by various methods

method / *Ns*	93	343	747
TSF	9.3610 e-03	3.9090 e-03	4.6700 e-04
MLS, n = 4	1.2385 e-07	3.6403 e-08	2.4867 e-08

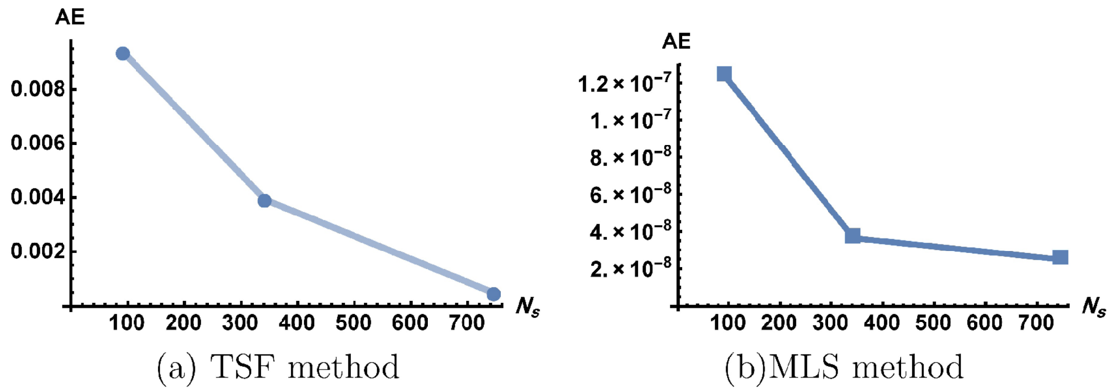


Fig. 3. Average Errors (AE) for various number of Ns in two methods (a) TSF method (b) MLS method for Example 4.6.

$$x(1-x)u_{xx}+y(1-y)u_{yy}=-4xy(1-x)(1-y), \qquad (x,y)\in\Omega \tag{4.9}$$

with condition

$$u(x,y)=0, \qquad\qquad (x,y)\in\partial\Omega$$

where $\Omega\cup\partial\Omega=[0,1]\times[0,1]$. *The exact solution of (4.6) respect to the given conditions, is*

$$u(x\,,y)=xy(1-x)(1-y).$$

We compare our proposed method with the other methods that are mentioned in [18] such as Ordinary Linear Approximation (OLA) and some radial basis functions methods respect to three radial basis functions, multi quadric (MQ), Linear Function (LF) and Thin-Plate Spline (TPS).

In [18] for radial basis functions approach, three radial basis functions are considered as follows:

- MQ: $R(r)=\sqrt{c^2+r^2}$;
- Linear functions (LF): $R(r)=r$;
- Thin-Plate Spline (TPS): $R(r)=r^2\ln r$.

ARE for various number of collocation point Ns are shown in Table 5, also in our proposed method MLS the N_s is the number of inner points in Ns that we approximate the solution Ω on those points and n=4.

Example 4.6. *In this example we consider another problem from [18] and compare the solutions in it with solutions of proposed MLS method. Let the following linear second order PDE are given*

$$u_{xx}+u_{yy}=-12xy, \quad (x\,,y)\in\Omega, \tag{4.10}$$

with the boundary condition

$$u(x\,,y)=0, \quad (x\,,y)\in\partial\Omega,$$

Where $\Omega\cup\partial\Omega=\{(x,y):x^2+y^2\le 1\}$ *the exact solution of (4.6) respect to the boundary conditions, is*

$$u(x;y)=xy(1-x^2-y^2).$$

In [18] M. Li et al. use Thin-Plate Spline (TPS) functions as radial basis functions and three nodal densities with node numbers $Ns=93$; 343 and 747. In this example we consider the same number of irregular distanced nodes as shown in Figure 2. For bivariate polynomials basis functions of degree at most n=4 and various numbers of Ns, AE are shown in Table 6 and Figure 3.

5. Conclusion

Comparing the proposed MLS method with some other existed methods demonstrate the accuracy and efficiency of our method. Another advantage of the MLS method is obtaining the approximation of the solution in any point of the given domain. In practice we often want to know the behavior of a system in some parts of its domain not on over whole of the domain, in the other words a local approximation of exact solution can be enough, for this purpose we can use the proposed method only for a given point x to achieve a local approximation of actual solution around this point. All kinds of linear PDEs can be solved by the proposed method.

მათემატიკა

წრფივი კერძოწარმოებულიანი დიფერენციალური განტოლებების ამოხსნა მოძრავი უმცირესი კვადრატების მეთოდით

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წინამდებარე სტატიაში განხილულია წრფივი კერძოწარმოებულიანი დიფერენციალური განტოლებების ამოხსნის მეთოდი, კერძოდ, სითბოსა და ტალღების ერთ ან მრავალგანზომილებიან გარემოში გავრცელება მოძრავ უმცირეს კვადრატთა მეთოდის გამოყენებით. მოცემულია რამდენიმე საილუსტრაციო მაგალითი და შემოთავაზებული მეთოდი შედარებულია სხვა მეთოდებთან მისი ეფექტიანობის თვალსაჩინოების მიზნით.

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