საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 9, №3, 2015
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 9, no. 3, 2015

Mathematics

On a Nonlocal Problem for an Abstract Ultraparabolic Equation

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ABSTRACT. In this paper ultraparabolic equation with nonlocal initial condition is considered in abstract Hilbert spaces. The existence and uniqueness result for the nonlocal problem is proved in suitable spaces of vector-valued distributions with values in Hilbert spaces. An iteration algorithm of approximation of solution of the nonlocal problem by a sequence of solutions of corresponding classical problems is constructed and investigated. Applying general result obtained for non-classical problem in abstract Hilbert spaces, nonlocal in time initial-boundary value problem for ultraparabolic equation is studied in Sobolev spaces. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *abstract ultraparabolic equation, nonlocal initial condition, initial-boundary value problem for ultraparabolic equation.*

Ultraparabolic equations and other evolution equations involving several time variables has various applications in physics (e.g., multi-parameter Brownian motion), biology (e.g., age structured population dynamics), mathematical finance and other areas of science. Various types of ultraparabolic problems have been studied (see [1-9] and references given therein), however non-classical problems for ultraparabolic equations with nonlocal conditions with respect to the time variables [10] are not widely investigated.

In the present paper we study ultraparabolic equation in abstract Hilbert spaces with two time variables, classical initial condition with respect to one time variable and nonlocal initial condition with respect to another time variable. We obtain existence and uniqueness result in spaces of vector-valued distributions with values in abstract Hilbert spaces, and prove that solution continuously depends on the given data in suitable spaces. We construct an iteration algorithm of approximation of solution of the nonlocal in time problem for ultraparabolic equation by classical problems, prove convergence of the sequence of solutions of the classical problems and estimate the rate of convergence. Moreover, we consider applications of the obtained general results to nonlocal in time problem for ultraparabolic partial differential equation with two time variables and general second-order elliptic operator.

We denote by $L(X;\,Y)$ the space of linear continuous operators from X to Y, where X and Y are Banach spaces. Let $C([0,T];X)$ denote the space of continuous vector-functions on $[0,T]$ with values in X. We denote by $L^2(\Delta;X)$ the space of such measurable vector-functions $g:\Delta\to X$ that $\|g\|_X\in L^2(\Delta)$ and the generalized derivatives of g we denote by $\partial g/\partial\tau_i\in D'(\Delta;X)=L(D(\Delta);X)$, $i=1,...,s$, $dg/d\tau_1=g'$, for $s=1$ (cf. [11]), where $D(\Delta)$ is the space of infinitely differentiable functions with compact support in Δ, where $\Delta\subset\mathbf{R}^s$, $s\in\mathbf{N}$ is a bounded domain.

Let V and H be separable Hilbert spaces, such that V is dense in H and continuously embedded in it. We identify space H with its dual by using the inner product $(.,.)$ in H and hence $V\subset H\subset V'$ with continuous and dense embeddings, where V' is the dual space of V. We denote by $\langle .,.\rangle$ the duality relation between V' and V. Let us consider the following space of vector-valued distributions

$$W=\{w\in L^2(\omega;V);\partial w/\partial t_1,\partial w/\partial t_2\in L^2(\omega;V')\},\ \ \omega=(0,T_1)\times(0,T_2),$$

which is a Hilbert space equipped with the norm

$$\|v\|_W=\left(\|v\|^2_{L^2(\omega;V)}+\|\partial w/\partial t_1\|^2_{L^2(\omega;V')}+\|\partial w/\partial t_2\|^2_{L^2(\omega;V')}\right)^{1/2}.$$

In the following two theorems we give some properties of the space W.

Theorem 1. *For each vector-function* $w\in W$ *and* $0\le\tau_1\le T_1$, $0\le\tau_2\le T_2$, *there exist traces* $w(\tau_1,t_2)=(tr_{t_1=\tau_1}w)(t_2)$ *and* $w(t_1,\tau_2)=(tr_{t_2=\tau_2}w)(t_1)$, *such that the trace operators* $tr_{t_1=\tau_1}:W\to L^2(0,T_2;H)$ *and* $tr_{t_2=\tau_2}:W\to L^2(0,T_1;H)$ *are continuous.*

Theorem 2. *For vector-functions* $w\in W$ *and* $v\in V$ *the following equalities are valid*

$$\left\langle\frac{\partial w}{\partial t_1},v\right\rangle=\frac{\partial}{\partial t_1}(w,v),\qquad\left\langle\frac{\partial w}{\partial t_2},v\right\rangle=\frac{\partial}{\partial t_2}(w,v)\ \text{ in }\ D'(\omega).$$

Let $A\in L(V;V')$ be a self-adjoint coercive operator, which means that the bilinear form $a(v,w)=\langle Av,w\rangle$ satisfies the conditions

$$\begin{aligned}&|a(v,w)|\le c_a\|v\|_V\|w\|_V,\\&a(v,w)=a(w,v),\quad a(v,v)\ge\hat{c}_a\|v\|_V^2,\end{aligned}\qquad\forall v,w\in V,\tag{1}$$

where $c_a,\hat{c}_a=const>0$. We also suppose that there exists a system of eigenvectors $\{v_k\}_{k\in\mathbf{N}}$ of the operator A corresponding to the eigenvalues $\{\lambda_k^2\}_{k\in\mathbf{N}}$, i.e. $a(v_k,v)=\lambda_k^2(v_k,v)$, for all $v\in V$, which is complete in V and orthonormal in H. From (1) it follows that the operator $A\in L(V;V')$ is invertible and $A^{-1}\in L(V';V)$ [12]. Note that $a(v_k,v)=\langle Av_k,v\rangle=\lambda_k^2(v_k,v)=\langle\lambda_k^2v_k,v\rangle$ and hence $Av_k=\lambda_k^2v_k\in V$, $v_k/\lambda_k^2=A^{-1}v_k$, for all $k\in\mathbf{N}$.

Let us define space $V_1=\{v\in V;Av\in H\}$, which is a Hilbert space equipped with the scalar product $(v,w)_1=(Av,Aw)$, for all $v,w\in V_1$. Space V_1 is continuously embedded in V, since

$$\|v\|_V=\left\|A^{-1}Av\right\|_V\le\left\|A^{-1}\right\|\|Av\|_{V'}\le c_1\left\|A^{-1}\right\|\|Av\|_H=c_2\left\|A^{-1}\right\|\|v\|_{V_1},\qquad c_1,c_2=const>0.$$

Note that the space V_1 is dense in V. Indeed, if $v\in V$, then $Av\in V'$. Since H is dense in V', there exists sequence $(h_n)_{n\ge1}$, $h_n\in H$, $n\in\mathbf{N}$, such that $h_n\to Av$ in V', as $n\to\infty$. Consequently, $A^{-1}h_n\to A^{-1}Av=v$ in V, as $n\to\infty$, and $A^{-1}h_n\in V_1$, $n\in\mathbf{N}$.

So, $V_1\subset V\subset H\subset V'\subset V_1'$ with continuous and dense embeddings, where V_1' is the dual space of V_1. The duality relation between V_1' and V_1 we denote by $\langle .,.\rangle_1$. Note that for all $v,w\in V_1$ we have

$$\left|\langle Av,w\rangle_1\right|=\left|(Av,w)\right|\le\|Av\|_H\|w\|_H=\|v\|_{V_1}\|w\|_H\le c_3\|v\|_{V_1}\|w\|_{V_1},\quad c_3=const>0,$$

and hence $\|Av\|_{V_1'}\le c_3\|v\|_{V_1}$. The restriction of the operator A on V_1 is a linear continuous operator from V_1 to

V_1' and since V_1 is dense in H, there exists unique continuous continuation of A on H.

In this paper we consider non-classical problem for an abstract ultraparabolic equation

$$\frac{\partial u}{\partial t_1}+\frac{\partial u}{\partial t_2}+Au=f, \qquad (t_1,t_2)\in(0,T_1)\times(0,T_2), \tag{2}$$

with the following classical and nonlocal initial conditions

$$u(0,t_2)=\varphi(t_2),\quad 0\le t_2\le T_2,\qquad u(t_1,0)=\alpha u(t_1,\xi)+\psi(t_1),\quad 0\le t_1\le T_1, \tag{3}$$

where $\alpha\in\mathbf{R}$, $0<\xi<T_2$, φ and ψ are given vector-functions from suitable spaces, which we define by applying Theorem 1, and A is a given linear continuous operator in corresponding spaces. In the present paper we investigate nonlocal problem (2), (3) applying the following variational formulation: find a vector-function $u\in W$, $\partial u(t_1,0)/\partial t_1\in L^2(0,T_1;V_1')$, $\partial u(0,t_2)/\partial t_2\in L^2(0,T_2;V_1')$, which satisfies the equation

$$\frac{\partial}{\partial t_1}\big(u(.,.),v\big)+\frac{\partial}{\partial t_2}\big(u(.,.),v\big)+a(u(.,.),v)=\big\langle f(.,.),v\big\rangle,\qquad \forall v\in V, \tag{4}$$

in the sense of distribution on ω, and initial conditions

$$\begin{aligned}&(tr_{t_1=0}u)(t_2)=\varphi(t_2), && 0\le t_2\le T_2,\\ &(tr_{t_2=0}u)(t_1)=\alpha(tr_{t_2=\xi}u)(t_1)+\psi(t_1), && 0\le t_1\le T_1,\end{aligned} \tag{5}$$

in the spaces $L^2(0,T_2;H)$ and $L^2(0,T_1;H)$, respectively. Applying Theorem 2 and by taking account of the density of linear combinations of products ϕv, $\phi\in D(\omega)$, $v\in V$, in $L^2(\omega;V)$ we obtain that equation (4) is equivalent to equation (2), which is considered as equation in the space $L^2(\omega;V')$.

The following existence and uniqueness theorem is valid for nonlocal problem (4), (5).

Theorem 3. *If $\varphi\in L^2(0,T_2;H)$, $\varphi'\in L^2(0,T_2;V_1')$, $\psi\in L^2(0,T_1;H)$, $\psi'\in L^2(0,T_1;V_1')$ and satisfy compatibility condition $\varphi(0)=\alpha\varphi(\xi)+\psi(0)$, and $f,\dfrac{\partial f}{\partial t_1},\dfrac{\partial f}{\partial t_2}\in L^2(\omega;V')$, then problem (4), (5) has a unique solution and the following estimate is valid*

$$\begin{aligned}\|u\|_W\le c\Big(&\|\varphi\|_{L^2(0,T_2;H)}+\|\varphi'\|_{L^2(0,T_2;V_1')}+\|\psi\|_{L^2(0,T_1;H)}+\|\psi'\|_{L^2(0,T_1;V_1')}\\ &+\|f\|_{L^2(\omega;V')}+\left\|\frac{\partial f}{\partial t_1}\right\|_{L^2(\omega;V')}+\left\|\frac{\partial f}{\partial t_2}\right\|_{L^2(\omega;V')}\Big).\end{aligned}$$

For nonlocal in time problem (4), (5) we can construct iteration algorithm of approximation of solution of the nonlocal problem by solutions of classical problems in corresponding spaces. We consider the following sequence of problems with classical initial conditions: find a vector-function $w_p\in W$, $\partial w_p(t_1,0)/\partial t_1\in L^2(0,T_1;V_1')$, $\partial w_p(0,t_2)/\partial t_2\in L^2(0,T_2;V_1')$, which satisfies the equation

$$\frac{\partial}{\partial t_1}\big(w_p(.,.),v\big)+\frac{\partial}{\partial t_2}\big(w_p(.,.),v\big)+a(w_p(.,.),v)=\big\langle f(.,.),v\big\rangle,\qquad \forall v\in V, \tag{6}$$

in the sense of distribution on ω, and initial conditions

$$\begin{aligned}&(tr_{t_1=0}w_p)(t_2)=\varphi(t_2), && 0\le t_2\le T_2,\\ &(tr_{t_2=0}w_p)(t_1)=\alpha(tr_{t_2=\xi}w_{p-1})(t_1)+\psi(t_1), && 0\le t_1\le T_1,\end{aligned}\qquad p\ge 1, \tag{7}$$

in the spaces $L^2(0,T_2;H)$ and $L^2(0,T_1;H)$, respectively, where $w_0\equiv 0$. From Theorem 3 we obtain that the classical problem (6), (7) has a unique solution for each $p\in\mathbf{N}$. In the following theorem we give result on the convergence of the sequence of vector-functions $(w_p)_{p=1}^{\infty}$ to the solution of the nonclassical problem.

Theorem 4. *If* $\alpha < 1$ *and* $\varphi \in L^2(0,T_2;H)$, $\varphi' \in L^2(0,T_2;V_1')$, $\psi \in L^2(0,T_1;H)$, $\psi' \in L^2(0,T_1;V_1')$, $\varphi(0) = \alpha\varphi(\xi) + \psi(0)$, $f, \frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2} \in L^2(\omega;V')$, *then problem* (4), (5) *has a unique solution, the sequence* $(w_p)_{p=1}^{\infty}$ *of solutions of problem* (6), (7) *tends to u in the space* $L^2(\omega;V)$ *and* $\left\|u - w_p\right\|_{L^2(\omega;V)} \le c\alpha^p$, $p \in \mathbf{N}$, $c = const > 0$.

Now let us consider application of the general results obtained for abstract nonclassical problem to nonlocal in time problem for ultraparabolic partial differential equation. Let $\Omega \subset \mathbf{R}^n$, $n \in \mathbf{N}$, be a bounded domain with Lipschitz boundary Γ [12]. By $H^k(\Omega)$ we denote the Sobolev space of the k-th order based on $L^2(\Omega)$, $k \in \mathbf{N}$. We denote the closure in $H^k(\Omega)$ of the set $D(\Omega)$ of infinitely differentiable functions with compact support in Ω by $H_0^k(\Omega)$ and its dual space we denote by $H^{-k}(\Omega)$. We consider the second-order elliptic operator

$$A \equiv -\sum_{q,j=1}^{n} \frac{\partial}{\partial x_q}\left(a_{qj}(x) \frac{\partial}{\partial x_j} \right) + a_0(x)$$

where $a_{qj}(x), a_0(x) \in L^\infty(\Omega)$, $q, j = 1,...,n$. We assume that the inequalities

$$a_0(x) \ge 0, \qquad \sum_{q,j=1}^{n} a_{qj}(x)\eta_j\eta_q \ge c\sum_{j=1}^{n}(\eta_j)^2, \quad c = const > 0, \tag{8}$$

are valid for almost all $x \in \Omega$, η_j ($j = 1,...,n$) are arbitrary real numbers. Partial derivatives in the definition of A are treated as generalized derivatives with respect to the corresponding variables. Note that the first-order partial derivative of a function in $L^2(\Omega)$ belongs to the space $H^{-1}(\Omega)$. Since the multiplication of a function in the space $L^2(\Omega)$ by a function in $L^\infty(\Omega)$ leaves it in $L^2(\Omega)$, for each function $v \in H_0^1(\Omega)$ we have $Av \in H^{-1}(\Omega)$. The elliptic operator A is a continuous operator from $H_0^1(\Omega)$ to the space $H^{-1}(\Omega)$, which, by virtue of (8), satisfies the following self-adjointness and coerciveness conditions

$$\langle Av, w \rangle_* = \langle Aw, v \rangle_*, \quad \langle Av, v \rangle_* \ge \alpha \| v \|_{H^1(\Omega)}^2, \qquad \forall v, w \in H_0^1(\Omega),$$

where $\alpha = const > 0$, $\langle .,. \rangle_*$ is the duality relation between the spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Note that $H_0^2(\Omega)$ is a dense subspace of $\{v \in H_0^1(\Omega); Av \in L^2(\Omega)\}$. The bilinear form $a(v,w) = \langle Av, w \rangle_*$ corresponding to the operator A is of the following form

$$a(v,w) = \int_\Omega \left(\sum_{q,j=1}^{n} a_{qj}(x) \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_q} + a_0(x) v w \right) dx$$

and satisfies conditions (1). Since the continuous embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact, it follows from properties of self-adjoint coercive operators mapping a Hilbert space into its dual [12] that there exists a system of orthonormal in $L^2(\Omega)$ and complete in $H_0^1(\Omega)$ eigenfunctions $\{v_k\}_{k=1}^{\infty}$ of the operator $A: H_0^1(\Omega) \to H^{-1}(\Omega)$ corresponding to the eigenvalues $\{\lambda_k^2\}_{k=1}^{\infty}$, such that $0 < \lambda_1 \le \lambda_2 \le \ldots$, $\lambda_k \to \infty$ as $k \to \infty$.

Let us consider nonclassical problem for ultraparabolic partial differential equation

$$\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} + Au = f(x,t_1,t_2), \qquad (x,t_1,t_2) \in \Omega \times (0,T_1) \times (0,T_2), \tag{9}$$

with the nonlocal initial and homogeneous boundary conditions

$$\begin{aligned} &u(x,0,t_2)=\varphi(x,t_2), \\ &u(x,t_1,0)=\alpha u(x,t_1,\xi)+\psi(x,t_1), \end{aligned} \qquad x\in\Omega, \tag{10}$$

$$u(x,t_1,t_2)=0, \quad (x,t_1,t_2)\in\Gamma\times(0,T_1)\times(0,T_2), \tag{11}$$

where Γ is the boundary of Ω, $\alpha\in\mathbf{R}$. Note, that vector-functions defined on $\Omega\times(0,T_1)\times(0,T_2)$ can be identified with vector-functions defined on $(0,T_1)\times(0,T_2)$ and ranging in the corresponding function spaces on Ω. Hence, the nonlocal in time problem (9)-(11) can be stated as the problem of finding a vector-function $u\in L^2(\omega;H_0^1(\Omega))$, $\partial u/\partial t_1,\partial u/\partial t_2\in L^2(\omega;H^{-1}(\Omega))$, which satisfies equation (9) in the space $D'(\omega;H^{-1}(\Omega))$ of distributions on ω ranging in $H^{-1}(\Omega)$, where the partial derivatives $\partial u/\partial t_1$, $\partial u/\partial t_2$ are treated as first-order generalized derivatives $\partial u/\partial t_1,\partial u/\partial t_2\in D'(\omega;H_0^1(\Omega))$. In addition, we seek for *u*, which satisfies conditions (10) in the spaces $L^2(0,T_2;L^2(\Omega))$ and $L^2(0,T_1;L^2(\Omega))$, respectively, and boundary conditions (11) are valid because the trace of vector-function from $H_0^1(\Omega)$ vanishes on the boundary of the domain Ω. Thus, the nonlocal in time problem (9)-(11) is a particular case of the nonclassical problem (4), (5), with $V=H_0^1(\Omega)$, $H=L^2(\Omega)$. The following existence and uniqueness theorem is valid for nonlocal in time problem (9)-(11).

Theorem 5. *If* $\varphi\in L^2(0,T_2;L^2(\Omega))$, $\varphi'\in L^2(0,T_2;H^{-1}(\Omega))$, $\psi\in L^2(0,T_1;L^2(\Omega))$, $\psi'\in L^2(0,T_1;H^{-1}(\Omega))$ *and satisfy compatibility condition* $\varphi(x,0)=\alpha\varphi(x,\xi)+\psi(x,0)$, *for almost all* $x\in\Omega$, *and* $f,\dfrac{\partial f}{\partial t_1},\dfrac{\partial f}{\partial t_2}\in L^2(\omega;H^{-1}(\Omega))$, *then problem* (9)-(11) *has a unique solution* $u\in L^2(\omega;H_0^1(\Omega))$, $\partial u/\partial t_1,\partial u/\partial t_2\in L^2(\omega;H^{-1}(\Omega))$ *and the following estimate is valid*

$$\begin{aligned} &\|u\|^2_{L^2(\omega;H_0^1(\Omega))}+\left\|\frac{\partial u}{\partial t_1}\right\|^2_{L^2(\omega;H^{-1}(\Omega))}+\left\|\frac{\partial u}{\partial t_2}\right\|^2_{L^2(\omega;H^{-1}(\Omega))}\le c\Big(\|\varphi\|_{L^2(0,T_2;L^2(\Omega))}+\|\varphi'\|_{L^2(0,T_2;H^{-1}(\Omega))} \\ &+\|\psi\|_{L^2(0,T_1;L^2(\Omega))}+\|\psi'\|_{L^2(0,T_1;H^{-1}(\Omega))}+\|f\|_{L^2(\omega;H^{-1}(\Omega))}+\left\|\frac{\partial f}{\partial t_1}\right\|_{L^2(\omega;H^{-1}(\Omega))}+\left\|\frac{\partial f}{\partial t_2}\right\|_{L^2(\omega;H^{-1}(\Omega))}\Big). \end{aligned}$$

მათემატიკა

აბსტრაქტული ულტრაპარაბოლური განტოლებისათვის არალოკალური ამოცანის შესახებ

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წარმოდგენილ ნაშრომში განხილულია ულტრაპარაბოლური განტოლება არალოკალური საწყისი პირობებით აბსტრაქტულ ჰილბერტის სივრცეებში. არალოკალური ამოცანისათვის დამტკიცებულია არსებობის და ერთადერთობის შედეგი ვექტორულ მნიშვნელობათა მქონე განაწილებების სივრცეებში მნიშვნელობებით ჰილბერტის სივრცეებში. აგებული და გამოკვლეულია არალოკალური ამოცანის ამონახსნის შესაბამისი კლასიკური ამოცანების ამონახსნების მიმდევრობით აპროქსიმაციის იტერაციული ალგორითმი. არაკლასიკური ამოცანისათვის აბსტრაქტულ ჰილბერტის სივრცეებში მიღებული ზოგადი შედეგის გამოყენებით სობოლევის სივრცეებში შესწავლილია დროით არალოკალური საწყის-სასაზღვრო ამოცანა ულტრაპარაბოლური განტოლებისათვის.

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Received November, 2015