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Mathematics

Configurations of Points as Coulomb Equilibria

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ABSTRACT. We discuss various aspects of a general problem naturally arising in the framework of an approach to inverse problems of electrostatics. In particular, we describe some developments in the spirit of the famous Maxwell conjecture on the number of equilibria in the case of three charges. Along the same lines we discuss the equilibrium configurations of charges confined to a system of concentric circles and their relation to reconfiguring of charged orbiting objects. We also outline possible applications of our approach to the electrostatic control of polygonal linkages. *© 2016 Bull. Georg. Natl. Acad. Sci.*

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1. Various topics concerned with equilibrium distributions of point charges with Coulomb interaction traditionally receive considerable attention (see, e.g., [1-3]). In particular, this refers to the longstanding Maxwell conjecture on the number of equilibria of a finite system of point charges in $\mathbf{R}^3$ [3]. Recently, a new direction of research within this circle of ideas was suggested in [4, 5]. Several later developments have been presented in [6-8]. The essential novelty of the approach developed in [4, 5] was that it focused on the so-called *inverse problem of electrostatics* (IPES). The present paper deals with an analogous problem in different setting called *electrostatic stabilization problem* (ESP).

The main idea is to represent a finite set of points in Euclidean space as an equilibrium configuration of a system of point charges with Coulomb interaction. To describe this setting more precisely we introduce the following terminology. Given a finite set (configuration) of points Z in three-dimensional Euclidean space $\mathbf{R}^3$ and a system S of n point charges in $\mathbf{R}^3$ such that all points of Z are critical points of the Coulomb potential of this system of point charges, we say that the system of charges S is a *Coulomb stabilizer* of Z. This concept naturally suggests the following general problem which is the organizing center of our considerations.

(ESP) Given a finite set of points Z in $\mathbf{R}^3$**,** investigate the existence and geometric structure of Coulomb stabilizers of Z.

Notice that a priori the set of Coulomb stabilizers of a given configuration Z may be empty. Our first main result (Theorem 1) establishes the existence of Coulomb stabilizers for any finite subset of $\mathbf{R}^3$. So it is

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meaningful to discuss the structure and possible applications of Coulomb stabilizers. The main aim of this note is to present several typical developments along these lines.

2. We begin with a precise description of the main objects and problems considered in the sequel. Let (P, Q) be a pair consisting of n-tuple of points $P = (p_1, \dots, p_n)$ of Euclidean space $\mathbf{R}^3$ and n-tuple of non-zero real numbers $Q = (q_1, \dots, q_n)$. Such a pair will be called a *configuration of point charges* and interpreted as a collection of point charges q_i placed at points p_i. To refer to this specific setting we use notation Q@P or Q/P. With this interpretation it is natural to introduce the *Coulomb potential* $E_{Q@P}$ of pair (P,Q) considered as a function on $\mathbf{R}^3$ defined by the usual formula

$$E_{Q@P}(z) = \sum \frac{q_i}{d_i},$$

where z is a point of $\mathbf{R}^3$ and d_i is the Euclidean distance between z and p_i. We will also consider the *resulting force* at point z defined as the gradient $\nabla E_{Q@P}$ at the point $z \in \mathbf{R}^3$ and equal to

$$F(z) = \sum F_i(z) = \sum \frac{q_i}{d_i^3}(z - p_i),$$

where the expression in each brackets should be understood as a vector equal to the difference of radius-vectors of the two points and d_i is the Euclidean distance between z and p_i

A critical point of Coulomb potential $E_{Q@P}$ is called an *equilibrium* of the configuration of charges Q@P. A classical fundamental problem of electrostatics is to find the equilibria of a given configuration of point charges Q@P and investigate the local behaviour of its Coulomb potential near to each equilibrium. In many cases all critical points of Coulomb potential are non-degenerate and then it is important to compute their Morse indices [1, 2].

Given a finite set (configuration) of m points Z in the plane or three-dimensional Euclidean space we say that a configuration of n point charges Q@P is a *Coulomb stabilizer* of Z if all points z_j are critical points of the Coulomb potential $E_{Q@P}$ of Q@P.

Theorem 1. Any finite set of points in $\mathbf{R}^3$ has a Coulomb stabilizer.

A complete proof of this theorem is rather involved so we present just the main idea and crucial steps which are in fact quite natural and simple. Notice first that finding Coulomb stabilizers naturally reduces to solving a system of equations in the coordinates of points p_i and values of charges q_i. The arising system of equations is quite complicated and it is unclear how to prove that it has real solutions. However this system is obviously linear in variables q_i which suggests the following strategy. Let us fix the positions of hypothetical stabilizing charges and search for their q_i. For fixed positions p_i, the denominators in the above expressions for the resulting force are the known numbers. So it remains to deal with a system of linear equations on q_i. If m is the number of points in Z and n is the number of sought charges then the condition that Z is an equilibrium is equivalent to 3m homogeneous linear equations in n variables q_i. Let us take $n = 3m + 1$ and add a normalizing equation $\Sigma\, q_i = 1$ in order to exclude trivial solutions to this system. This yields a system of 3m+1 linear equations in 3m+1 variables which is already non-homogeneous. To complete the proof it is sufficient to show that we can choose the points p_i in such way that the determinant of this system is non-vanishing. Since this determinant is a non-vanishing rational expression in coordinates of points p_i, it should be non-zero for a generic collection of points p_i. In other words, for almost all choices of 3m+1 points p_i this determinant does not vanish so the extended linear system for q_i has a non-trivial solution as was claimed. Now it is not difficult to obtain a rigorous proof using some standard tools from topology and algebraic geometry.

Remark 1. This result and the method of proof remain valid for a wide class of other interactions defined by so-called *central forces*. Examples include logarithmic potential and Coulomb like potentials with arbitrary degree $d > 1$ of distances d_i in the denominator. Consequently, much of the following discussion can be generalized to such potentials. One can treat in the same way configurations in Euclidean spaces of any dimension and differential submanifolds of Euclidean spaces. Generalizations of such type will be discussed elsewhere.

Remark 2. Analyzing the above argument it is easy to see that any planar finite configuration of points can be stabilized by a configuration of point charges lying in the same plane. Analogously, if a configuration is aligned, i.e. all of its points are collinear, then it can be stabilized by a configuration of charges lying on the same line.

We continue by discussing further issues concerned with Coulomb stabilizers. For any finite subset Z, it is interesting to find the minimal cardinality of Coulomb stabilizers for Z which will be called the *Coulomb index* of Z and denoted $ind_c(Z)$. A number of natural questions are concerned with the concept of Coulomb index. Let us call a stabilizer *minimal* if it contains exactly $ind_c(Z)$ charges. One may now wish to develop methods for calculating $ind_c(Z)$ and describing the structure of the totality of minimal Coulomb stabilizers for Z.

Next, let us say that Q@P is an *exact stabilizer* of Z if the set of critical points of $E_{Q@P}$ coincides with Z (in other words, there are no other critical points outside Z). It remains unclear if any finite set has an exact Coulomb stabilizer. We suspect that this is not true and then one should try to characterize configurations admitting exact stabilizers. It is also interesting to characterize configurations admitting exact minimal stabilizers.

Furthermore, in various situations it is desirable to consider equilibria which are in some sense stable. Some issues of such kind have been discussed in [6, 8]. Often stable equilibrium is defined as non-degenerate local minimizer of Coulomb energy. Consequently, given a stabilizer of configuration Z it is natural to find out which points of Z are local minima. A more general problem in this spirit is to find conditions which guarantee that all points in Z are non-degenerate critical points of Coulomb potential.

With this in mind, we say that a configuration of charges is a *Coulomb trap* for configuration Z if all points of Z are non-degenerate local minimizers of Coulomb potential. An appropriate modification of the proof of Theorem 1 shows that Coulomb traps exist for any finite subset of Euclidean space. Let us denote by $ind_t(Z)$ the minimal cardinality of Coulomb traps for Z. This suggests a few problems concerned with $ind_t(Z)$. Simple examples show that $ind_t(Z)$ does not in general coincide with $ind_c(Z)$. Is it possible to give an exact estimate of $ind_t(Z)$ in terms of $ind_c(Z)$? It seems intuitively clear that there should exist Coulomb traps with all charges of the same sign. Is it true that each configuration has a Coulomb trap with stabilizing charges of the same sign?

One can easily formulate many further problems concerned with the above concepts. These and similar issues can be informally referred to as *ESP paradigm*. Notice a conceptual analogy with the IPES paradigm considered in our previous papers [4, 6, 7]. However the two settings are in fact essentially different and constitute two independent topics.

It is remarkable that both these settings can be used for controlling the shapes of configurations by changing the values of stabilizing charges. Using Coulomb traps one may develop various scenarios of robust control of configurations. Such scenarios and related results can be informally called the *Coulomb equilibristics*. Coulomb equilibristics in the context of IPES has been discussed in [7, 8]. In the sequel we present examples of such kind arising in the context of ESP paradigm.

3. We proceed by discussing Coulomb stabilizers for special classes of configurations and their relations

to the famous *Maxwell conjecture on equilibria of point charges* [3] in the case of three charges. Namely, we study in some detail the Coulomb stabilizers of configurations in $\mathbf{R}^2$ containing not more than four points.

Recall that the aforementioned *Maxwell conjecture on equilibria of point charges* (MCEPC) states that if the set of equilibria of n point charges with Coulomb interaction in $\mathbf{R}^3$ is finite then their number does not exceed $(n-1)^2$. Remarkably, this conjecture remains unproven even for n=3 (see, e.g., [3]). Notice that the concept of Coulomb index of point configuration suggests the following problem which is similar in spirit to MCEPC and will be called *Coulomb index problem* (CIP) for point configurations.

(CIP) What is the minimal number I(m) such that any configuration of m points has a Coulomb stabilizer with no more than I(m) charges?

An analogous problem can of course can be formulated for Coulomb traps.

MCEPC and the results obtained for planar configurations with a few points suggest the following conjecture which can be considered as a converse to MCEPC in the planar case.

Conjecture 1. Any planar set consisting of m points has a Coulomb stabilizer with no more than [m/2] + 1 charges, where square brackets denote the integer part of m/2.

In the sequel we only consider planar configurations and charges of the same sign which will be referred to as *repelling charges*. This simplifies some formulations and permits to avoid complications connected with Earnshaw theorem. So in the rest of the paper we assume that all charges are positive.

Let us consider first a few cases with small m. For m=1, one point can be stabilized by two charges placed so that the three points are collinear. This is obviously the minimal number so $ind_c(\{pt\}) = 2$. Coulomb trap for one point cannot consist of two charges since they cannot prevent displacements transversal to the line through these points. Evidently, there exist traps with three charges so $ind_t(\{pt\}) = 3$. All other issues mentioned above have in this case trivial solutions as well.

For m=2, most of these problems also have easy solutions.

Proposition 1. Let |Z|=2. Then $ind_c(Z) = 2$ and Z can be stabilized by two charges belonging to the line connecting the points in Z. The set of minimal stabilizers is two-dimensional and they can be expressed by explicit formulae. Minimal stabilizer is exact if both of its charges are of the same sign and then both points in Z are stable equilibria of E so $ind_t(Z)=2$.

For |Z|=3, it is reasonable to consider two cases: the three points in Z are collinear (aligned configuration). or they form a non-degenerate triangle (a generic configuration). Without loss of generality we may assume that Z = {(1,0), (0,1), (a,b)}. The solutions to the above problems can be formulated as follows.

Proposition 2. Let |Z|=3. Then $ind_c(Z) \leq 4$. If Z is aligned then it can be stabilized by two or three charges belonging to the line containing Z. If Z is generic then it can be stabilized by three point charges.

The proof is based on the following observations. Denote by Q@P the sought system of stabilizing charges for Z. Let $(P,Q) = \{(x_j, y_j, q_j)\}$, where q_j are the values of charges. The fact that Q@P stabilizes three points is equivalent to 6 equations in nine parameters (x_j, y_j, q_j). Taking into account that stabilizing charges are only defined up to a common real factor we conclude that the number of effective parameters equals 8. So it is intuitively clear that such a system should generically have two-dimensional set of solutions. To prove that this is really the case one shows that the rang of (the Jacobian matrix) of this system is generically equal to six. Then its solvability can be proven using homotopy argument.

In many case the stabilizing charges can be chosen to belong to a circle centered at the barycenter of Z. Having in mind further analysis of Maxwell conjecture it would be interesting to characterize such configurations Z. Counting of parameters suggests that their set should be one-dimensional. We wonder if it can be given by an explicit equation on parameters (a, b).

Let now $|Z|=4$. Then it is natural to consider three cases: all points are collinear (aligned), exactly three points are collinear (quasi-aligned), no triple of points is collinear (generic). In first two cases it is easy to show that Z can be stabilized by four charges. In generic case we have eight equations for eight effective variables and there is no general reason that such a system of equations has a real solution and in fact we were unable to prove that Z can be stabilized by four charges. However it is possible to prove that Z can be stabilized by five point charges. In some cases where Z has certain symmetry it still can be stabilized by four charges. In the special case of aligned configuration we have a reasonable estimate for I(m).

Theorem 2. Let Z be an aligned configuration consisting of m points. Then it has a Coulomb stabilizer with no more than $m + 1$ charges belonging to the same line.

The proof is similar to the proof of Theorem 1. We choose m+1 points p_i interlacing with m given points. In this case one can write the arising linear system in explicit form and express its determinant in terms of distances d_{ij}. Then it is easy to conclude that this determinant does not vanish for a generic collection of interlacing points.

There are examples showing that the value of $ind_c(Z)$ depends on the distances between the collinear points. It is unclear how to give an explicit formula or effective algorithm for computing $ind_c(Z)$ even in the case of aligned configuration.

Let us now show how the above considerations can be used to reach some progress with Maxwell conjecture. We illustrate this in the case of three charges. Then it is known that all equilibria of their Coulomb potential belong to the plane containing these charges. This implies that Maxwell conjecture would be established if we were able to prove that no quintiple of points in the plane can be stabilized by three charges. To show that the latter fact is highly plausible let us count parameters. Denote again by Z the given configuration and by Q@P the sought system of stabilizing charges for Z. Let $(P,Q) = \{(x_j, y_j, q_j)\}$, where q_j are the values of charges. The fact that Q@P stabilizes five points is generically equivalent to 10 equations on nine parameters (x_j, y_j, q_j). So it is intuitively clear that such a system should generically have no solutions.

This gives an informal (but convincing) evidence that Maxwell conjecture is true for three charges. We believe that a rigorous proof can be achieved along these lines by a thorough analysis of the arising system of equations using computer algebra packages. However we did not manage to complete the argument.

One can explicate Theorem 2 in many concrete cases and calculate $ind_c(Z)$ for $|Z| \leq 6$. Those results give strong evidence in favour of the following conjecture.

Conjecture 2. Any aligned configuration containing $m \geq 2$ points can be stabilized by $[m/2]+1$ charges lying on the same line.

Summing up, there is considerable information about the ESP in the case of aligned configurations. In the next section we present some related results for another special class of configurations and indicate their possible application to the problem of reconfiguring charged geostationary satellites described in [9].

4. We consider now a system of n concentric circles $C = \{C_j\}$ in the plane and configurations Z of n points such that there is exactly one point of Z on each of the circles C_j. Our aim in this section is to find values of n charges $Q = \{q_j\}$ such that the configuration of charges Q@Z stays in rest assuming that the only forces are those of Coulomb interaction and each charge can only move along the circle which contains it. In such case we say that Z is a *Coulomb equilibrium with respect to C*. In this setting the main role is played by the Coulomb energy of Q@Z configuration.

Recall that the *Coulomb energy* of configuration of n point charges Q@Z (up to a constant multiple which we omit as irrelevant for our considerations) is defined as

$$E_{Q/P} = \sum_{i<j} \frac{q_i q_j}{d_{ij}},$$

where d_{ij} is the distance between points p_i and p_j. As is well known, the resulting force acting on q_i in position p_i (the gradient of potential $E_{Q@P}$ at point p_i multiplied by q_i) is equal to

$$F_i = \sum F_{ji} = \sum \frac{q_i q_j}{d_{ij}^3}(p_i - p_j),$$

where p_i denotes the radius-vector of point p_i and $F_{ji} = \sum \frac{q_i q_j}{d_{ij}^3}(p_i - p_j)$ is the electrostatic force (under our assumption it is repelling) acting on q_i at p_i due to its Coulomb interaction with q_j at p_j.

Thus, the fact that Z is a Coulomb equilibrium with respect to C is equivalent to requiring that Z is a constrained critical point of $E_{Q/P}$. In such a case we say that collection of charges Q is *stationary* for Z. In our case it means that, for each point of Z, the resulting force should be orthogonal to the corresponding circle. As in the previous section our main aim is to investigate and geometrically characterize those configurations Z for which there exists a collection of stationary charges Q.

The first non-trivial case of three charges on three concentric circles C_j has been discussed in big detail in [6]. The main result of this section generalizes an analogous result in [6]. We say that configuration Z on C is *balanced* if no closed half-plane contains all points of Z. Configuration Z is called *non-degenerate* if no diameter of the biggest circle contains more than one point of Z.

Theorem 3. If n is odd then any non-degenerate n-tuple of points Z on a system of n concentric circles is a Coulomb equilibrium of positive point charges. A non-degenerate set of n points is a stable Coulomb equilibrium if and only if it is balanced.

The proof follows the same lines as in [6]. It is based on the analysis of the matrix expressing the fact that the gradient of Coulomb energy is orthogonal to each of the circles C_j at point p_j. This condition can be rewritten as a linear system on values of sought charges q_j. It turns out that the matrix of this system can be factored in product of two matrices one of which is skew-symmetric. Since the dimension of matrix is odd its determinant vanishes, which implies that the linear system for values of charges has non-trivial solutions. The non-degeneracy condition implies that there are solutions with all components positive which gives the desired stabilizing charges. The fact that Z is balanced implies that Z is a non-degenerate local minimum of the Coulomb energy.

One of the possible applications of this result is related to the problem of reconfiguring swarms of geostationary charged satellites [9]. One may think of such a swarm as a configuration of the type considered in this section. The fact that it can be stabilized by values of charges means that change of those charges will force the swarm to take the shape of configuration minimizing the Coulomb energy. In other words, it is possible to control the configuration of swarm by properly choosing the values of stabilizing charges. This idea can be realized in various scenarios some of which were used in the case of three satellites [9]. Theorem 3 enables one to generalize results of [9] to the case of swarm consisting of more than three satellites.

It is interesting to develop Morse theory, in particular give index formulae for the Coulomb energy in general case of n concentric circles. The same refers to the signed area A and perimeter of configuration. Aligned configurations play special role since they are critical in all cases. In this context aligned configurations can be called *parades of (orbiting) points* by the way of analogy with parades of planets in astronomy. For A and P, Morse indices of parades can be easily computed from their combinatorics. It would be interesting to do the same for the Coulomb energy. Needless to say, all these problems are meaningful for a system of non-intersecting closed loops in $\mathbf{R}^3$. The case of non-intersecting ellipses can serve as a useful model of some real situations encountered in astronomy.

5. In conclusion we outline another possible application of our paradigm in the spirit of control theory. This is related to the Coulomb control of polygonal linkages discussed in [7]. Recall that the setting of [7] was concerned with a planar polygonal linkage with point charges placed at its vertices. For brevity we only discuss the case of quadrilateral linkage L with pairwise non-equal lengths of links. Such linkage has a smooth one-dimensional configuration space M(L) and one can consider Coulomb potential of charges v_i placed at its vertices as a function on M(L). It was shown in [7] that each convex configuration of L is a non-degenerate global minimum of Coulomb potential for certain values of vertex charges v_i. As was explained in [7], this yields an algorithm of robust complete control of convex configurations of L by values of vertex charges.

The approach accepted in this paper suggests that one may also try to control configurations of L by the charges q_j placed at fixed points p_j in the reference plane. This may be useful for some situations arising in the control of nano-particles and small autonomous robots. For simplicity let us assume that all vertex charges are equal to one. Then we may calculate the Coulomb energy E of the whole system of point charges $(v_i; q_j)$ and consider it as a function on M(L). It is natural to assume that the linkage will take the shape with minimal Coulomb energy. In this setting, the first step towards developing Coulomb control is to find conditions under which each configuration of L is an equilibrium of E|M(L). These conditions should be formulated in terms of the configuration of charges Q@P.

Using considerations and results of previous sections it is possible to show that this can be guaranteed if the number of controlling charges is four and they are situated at the vertices of a sufficiently big square surrounding the work-space of L. Moreover, one can explicitly calculate the values of stabilizing charges from the shape of a equilibrium-to-be configuration. So in the case of quadrilateral linkage this scenario of Coulomb control can be described rigorously and explicitly. Obviously, an analogous scenario may be considered for polygonal linkage with arbitrary number of links, which suggests a lot of open problems in the spirit of the main topics of this paper.

მათემატიკა

წერტილების კონფიგურაციები როგორც კულონის პოტენციალის წონასწორობები

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განხილულია ზოგადი ამოცანა, რომელიც დაკავშირებულია ელექტროსტატიკის შექცეულ ამოცანასთან. კერძოდ, მიღებულია რამდენიმე შედეგი მაქსველის ჰიპოთეზის შესახებ. შესწავლილია ელექტროსტატიკური წონასწორობები იმ სიტუაციაში, როდესაც მუხტები განლაგებულია ჩადგმული წრეწირების სისტემაზე. აღწერილია აგრეთვე მიღებული შედეგების შესაძლო გამოყენებები სახსრული მრავალკუთხედების მართვის ამოცანებში.

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