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Mathematics

The Application of Fuzzy Systems to Approximate the Solutions of a Nonlinear System of Equations

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ABSTRACT. In this paper, systems of nonlinear equations is considerd to solve. To approximate the solution we are interested in combining both fuzzy system and itterative methods. Elementary properties of designed fuzzy system are given. To design a fuzzy systems we used singleton fuzzifier, and center average defuzzifier. The proposed method overcomes the difficulties that arising in calculating complicated system of nonlinear equations using different itterative methods. To show the accuracy of the method, we apply it to some examples including some types of nonlinear systems of equations. *© 2016 Bull. Georg. Natl. Acad. Sci.*

Keywords: Fuzzy systems; systems of nonlinear equations; Center average defuzzifier; Iterative methods, Approximated solution

1. Introduction

Many problems in various areas led to the solution of linear and nonlinear systems of equetions. Systems of simultaneous nonlinear equations play a major role in mathematics and engineering. To solve such kind systems many methods have been proposed such as Newton method, Halley's method, Secant method and Broyden method [1-6]. To obtaine the solution of obtimal control problems, such as the global control of nonlinear diffusion equations [7, 8], and to solve optimal shape design and linear and nonlinear ODE's and inifinite-horizon optimal control problems [9, 10, 11], measure theory method used. And also some numerical methods applied by researchers to solve system of nonlinear equations such as Adomian's [12, 13]. In this paper, we are interested in combining both fuzzy system and iterative methods such as Newton's. Let's consider a system of the nonlinear equations as the following form [14]:

$$\begin{cases} f_1(x_1, x_2, \dots, x_n) = 0, \\ f_2(x_1, x_2, \dots, x_n) = 0, \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) = 0 \end{cases} \tag{1}$$

where f_i, $i = 1, 2, \ldots, n$ are nonlinear function in $U_i \in \mathbb{R}^n$, where $U = \cap_{i=1}^{n} U_i$. The vectorial function $F(x_1, x_2, \ldots, x_n) = [f_1(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n)]^T$ is a function in $U \subseteq \mathbb{R}^n$, by considering the unknown vector $x = [x_1, x_2, \ldots, x_n]^T$ Eq.(1) will be $F(x) = 0$. Up to now, vast investigations about the solution of nonlinear equations and nonlinear systems is done by researchers [14, 15, 16]. The most important application of fuzzy systems have concentrated on control problems because they can be used as open-loop controllers or closed-loop controllers which have applications in consumer electronics and industrial processes, respectivly. For example, fuzzy washing machines, fuzzy systems in cars, digital image stabilizer, and etc. Fuzzy systems are systems to be precisely defined. That is the theory of fuzzy systems itself is precise. There are tree types of fuzzy systems [17, 18]: (i) pure fuzzy systems, (ii) Takagi-Sugeno-Kang (TSK) fuzzy systems, and (iii) fuzzy systems with fuzzifier and defuzzifier. Inputs and outputs in pure fuzzy systems are fuzzy sets and it’s the main problem of this kind of systems, whereas they are real-valued variables in engineering systems. TSK proposed fuzzy system whose inputs and outputs are real-valued variables. However this system has its disadvantages as follows: Its THEN part is a mathematical formula so may not be able to represent human knowledge, and the versatility of fuzzy systems is not well-represented in this structure because there is not much freedom left to apply different principles in fuzzy logic. In order to solve this problem, the third type of fuzzy systems with fuzzifier and difuzzifier has been introduced. This fuzzy system overcomes mentioned disadvantages of the previous fuzzy systems.

To achieve the aim of solving nonlinear system of equetions, we design the fuzzy system $f(x)$ from the M fuzzy IF-THEN rules using product inference engine, singleton fuzzifier, and center average defuzzifier. The structure of fuzzy system is shown in Fig. 1. The rest of this paper is organized as follows: In section 2, we introduce some elementary properties of designed fuzzy system and discuss basic definitions. In Section 3 we study in detail the design of fuzzy system with singleton fuzzifier, inference engine and defuzzifier. Approximation accuracy of fuzzy systems will be shown in this section and some examples show the efficiency of the method in Section 4. Conclusion is drawn in Section 5.

2. Preliminaries

In this section we recall some basic notations which is used in fuzzy systems. We begin with defining the fuzzy rule.

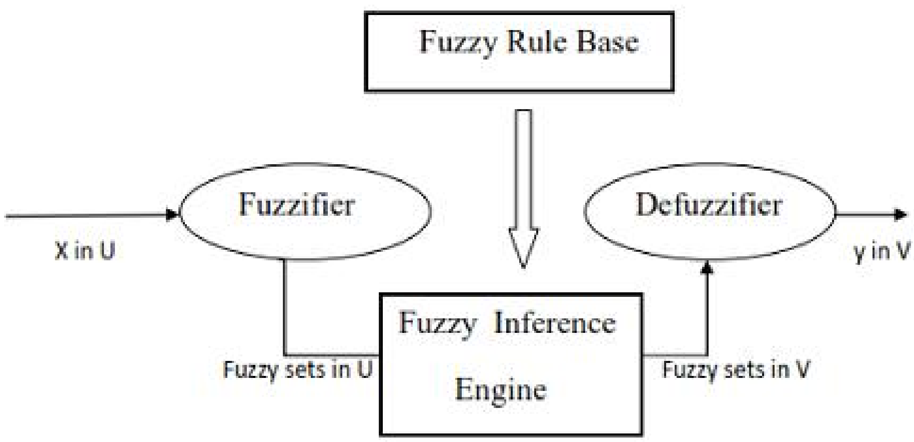

Fig. 1. Basic configuration of fuzzy systems with fuzzifier and defuzzifier.

2.1. Fuzzy Rule

A fuzzy rule base consists of a set of fuzzy IF-THEN rules. Consider the fuzzy system where $U = U_1 \times U_2 \times \ldots \times U_n \subset \mathbb{R}^n$ and $V \subset \mathbb{R}$. The fuzzy rule base comprises the following fuzzy IF-THEN rules:

$$Ru^l : If\ x_1\ is\ A_1^l, x_2\ is\ A_2^l, \ldots, x_n\ is\ A_n^l\ Then\ y\ is\ B^l \tag{2}$$

where A_i^l and B^l are fuzzy sets in $U_i \subset \mathbb{R}$ and $V \subset \mathbb{R}$, respectivly. Also $x = (x_1, x_2, \ldots, x_n)^T \in U$ and $y \in V$ are the input and output variables of the fuzzy system, respectively.

Definition 1. A set of fuzzy IF-THEN rules is complete if for any $x \in U$, there exists at least one rule in the fuzzy rule base, say rule Ru^l , such that

$$«_{A_i^l}(x) \neq 0 \tag{3}$$

for all $i = 1, 2, \ldots, n$.

Definition 2. A set of fuzzy IF-THEN rules is consistent if there are no rules with the same IF parts and different THEN parts.

Definition 3. A set of fuzzy IF-THEN rules is continuous if there do not exist such neighboring rules whose THEN part fuzzy sets have empty intersection.

Definition 4 (Pseudo-Trapezoid). Let $[a,b] \subset \mathbb{R}$. The Pseudo-Trapezoid membership function of fuzzy set A is a continuous function in $\mathbb{R}$ given by

$$«_A(x; a,b,c,d,H) = \begin{cases} I(x), & x \in [a,b), \\ H, & x \in (b,c], \\ D(x), & x \in (c,d], \\ 0, & x \in \mathbb{R} - (a,d) \end{cases} \tag{4}$$

2.2. Fuzzy Inference

In a fuzzy inference engine, fuzzy logic principles are used to combine the fuzzy IF-THEN rules in the fuzzy rule base into a mapping from a fuzzy set A' in U to a fuzzy set B' in V. Here, the Lasens max-product inference is considered [19].

Definition 5 (Product Inference Engine). In product inference engine, we use individual rule based inference with union combination, Mamdani's product implication, and algebraic product for all the t-norm operators and max for all the s-norm operators. The product inference engine can be written as follows:

$$«_{B'}(y) = \max_{l=1}^{M}[\sup(«_{A'}(x)\prod_{i=1}^{n} «_{A_i^l}(x_i)«_{B^l}(y))] \tag{5}$$

That is, given fuzzy set A' in U, the product inference engine gives the fuzzy set B' in V according to (4).

2.3. Defizzification Method

The defuzzifier can be defined as a mapping from fuzzy set B' in $V \subset \mathbb{R}$ (which is the output of the fuzzy inference engine) to crisp point $y^* \in V$. The output of the inference process is a fuzzy set, Generally, center-of-gravity method for defuzzification is used to fuzzy systems . The center of gravity defuzzifier specifies the y^* as the center of the area covered by the membership function of B' , that is [20, 17, 18]

$$y^* = \frac{\int_V y\mu_{B'}(y)dy}{\int_V \mu_{B'}(y)dy}, \tag{6}$$

where $\int_V$ is the conventional integral. If we view $\mu_{B'}(y)$ as the probability density function of a random variable, then the center of gravity defuzzifier gives the mean value of the random variable. Because the fuzzy set B' is the union or intersection of M fuzzy sets, an acceptable approximation of Eq. (2) is the weighted average of the centers of M fuzzy sets, with the weights equal the heights of the corresponding fuzzy sets.

Definition 6 (Center Average Defuzzifier). The fuzzy set B' is the union or intersection of M fuzzy sets. Let $\overline{y}^l$ be the center of the l'th fuzzy set and w_l be its height, the center average defuzzifier determines y^* as:

$$y^* = \frac{\sum_{l=1}^{M} \overline{y}^l w_l}{\sum_{l=1}^{M} w_l}. \tag{7}$$

Definition 7 (Singleton fuzzifier). The singleton fuzzifier maps a real-valued point $X^* \in U$ into a fuzzy singleton A' in *U* which has membership value 1 at X^* and 0 all other points in *U*; that is,

$$\mu_{A'}(X) = \begin{cases} 1 & if \;\; X = X^* \\ 0 & other\ wise, \end{cases} \tag{8}$$

Definition 8 (Completeness of Fuzzy Sets). Fuzzy sets $A^1, A^2, \ldots, A^N$ in $U \in \mathbb{R}$ are said to be complete on *U* if for any $x \in U$, there exists A^j such that $\mu_{A^j}(x) > 0$.

Definition 9 (Order Between Fuzzy Sets). For two fuzzy sets *A* and *B* in $W \subset \mathbb{R}$, we say $A > B$ if $hgt(A) > hgt(B)$ (that is, if $x \in hgt(A)$ and $x' \in hgt(B)$, then $x > x'$).

Lemma 1. *Suppose that the fuzzy set* B^l *is normal with center* $\overline{y}^l$*, then the fuzzy system with fuzzy rule base product inference engine, singleton fuzzifier, and center average defuzzifier is the following:*

$$k(x) = \frac{\sum_{l_1=1}^{M} \overline{y}^l \prod_{i=1}^{n} \mu_{A_i^l}(x_i)}{\sum_{l_1=1}^{M} \prod_{i=1}^{n} \mu_{A_i^l}(x_i)} \tag{9}$$

$x \in U \subset \mathbb{R}^n$ *is the input to the fuzzy system, and* $k(x) \in V \subset \mathbb{R}$ *is the output of the fuzzy system.*

Proof. See [20].

3. Design of Fuzzy System

To design a fuzzy system that have some nice properties and notational simplicity, we consider a fuzzy system with *n* inputs as $(x_1, \ldots, x_n)$. $F(x_1, \ldots, x_n)$ will be vectorial function in $U = U_1 \times \ldots U_n \in \mathbb{R}^n$. As it is mentioned in (2), fuzzy IF-THEN rule is interpreted as a fuzzy relation in the input-output product space $U \times V$ as follows:

$$Ru^{(l)} : A_1^l \times \ldots \times A_n^l \to B^l, \quad l = 1, 2, \ldots, N.$$

while $A_1^l \times \ldots \times A_n^l$ is a fuzzy relation in U defined by

$$\text{«}_{A_1^l \times \ldots \times A_n^l}(x_1, \ldots, x_n) = \text{«}_{A_1^l}(x_1) * \ldots * \text{«}_{A_n^l}(x_n), \quad l = 1, 2, \ldots, N.$$

In this study we used product inference engine as it is defined in Definition 5. Finally, by center average defuzzifier (9), we can approximate the recursive Newton's formula which is constructed from vectorial function $F(x_1, \ldots, x_n)$. This purpose, consider

$$G = X - [JF(X)]^{-1} F(X), \qquad X = (x_1, \ldots, x_n).$$

We will design a defuzzifier like k to approximate G. Let G be a function on closed an bounded set $U = [\sim_1, \ _1] \times [\sim_2, \ _2] \times \ldots \times [\sim_n, \ _n] \subset \mathbb{R}^n$ and G be unknown function. The target is design a fuzzy system to approximate G. To obtain the solution of $F(x) = 0$ every iterative method such as Newton method, Fix point method and etc. can be applied. In this study, an operator P transforms the function F to another function such G.

Lemma 2 (see [20]). *If* $A^1, A^2, \ldots, A^N$ *are consistent and normal fuzzy sets in* $U \in \mathbb{R}$ *with pseudo-trapezoid membership functions* $\text{«}_{A^j}(x; a_i, b_i, c_i, d_i)$ *for* $i = 1, 2, \ldots, N$, *then there exists a rearrangement* $\{1, 2, \ldots, N\}$ *of* $\{i_1, i_2, \ldots, i_N\}$ *such that*

$$A^{i_1} < A^{i_2} < \ldots < A^{i_N}. \tag{10}$$

Proof. For arbitrary $i, j \in \{1, 2, \ldots, N\}$, it must be true that $[b_i, c_i] \cap [b_j, c_j] = \text{¨}$, since otherwise the fuzzy sets $A^1, A^2, \ldots, A^N$ would not be consistent. Thus, there exists a rearrangement $\{i_1, i_2, \ldots, i_N\}$ of $\{1, 2, \ldots, N\}$ such that

$$[b_{i_1}, c_{i_1}] < [b_{i_2}, c_{i_2}] < \ldots < [b_{i_N}, c_{i_N}] \tag{11}$$

which implies (9).

Consider the following steps:

• Define N_i $(i = 1, 2, \ldots, n)$ fuzzy sets $A_i^1, A_i^2, \ldots, A_i^{N_i}$ in $[\sim_i, \ _i]$ which are normal, consistent, complete with pesudo-trapezoid membership functions $\text{«}_{A_i^1}(x_i; a_i^1, b_i^1, c_i^1, d_i^1), \ldots, \text{«}_{A_i^{N_i}}(x_i; a_i^{N_i}, b_i^{N_i}, c_i^{N_i}, d_i^{N_i})$ and $A_i^1 < A_i^2 < \ldots < A_i^{N_i}$ where $e_1^1 = \sim_1, e_1^{N_1} = \ _1, e_1^j = 1/2(b_1^j + c_1^j)$ for $j = 1, 2, \ldots, N_1 - 1$. Similarly we have $e_2^1 = \sim_2, e_2^{N_2} = \ _2, e_2^j = 1/2(b_2^j + c_2^j)$ for $j = 1, 2, \ldots, N_2 - 1$ and $e_n^1 = \sim_n, e_n^{N_n} = \ _n, e_n^j = 1/2(b_n^j + c_n^j)$ for $j = 1, 2, \ldots, N_n - 1$.

• Let $M = N_1 \times N_2 \times, \ldots, \times N_n$, so the fuzzy IF-THEN rule has the form of the following:

$Ru^{i_1 i_2 \ldots i_n}$: If x_1 is A_i^1, x_2 is A_i^2, ..., x_n is A_i^n. Then y is $B^{i_1 i_2 \ldots i_n}$,
where $i_1 = 1, 2, \ldots, N_1, i_2 = 1, 2, \ldots, N_2$ to $i_n = 1, 2, \ldots, N_n$.

The center of the fuzzy set $B^{i_1 i_2 \ldots i_n}$, denoted by $\bar{y}^{i_1 i_2 \ldots i_n}$, is chosen as

$$\bar{y}^{i_1 i_2 \ldots i_n} = g(e_1^1, e_2^2, \ldots, e_n^n). \tag{12}$$

• Construct the fuzzy system $f(x)$ from the $N_1 \times N_2 \times, \ldots, \times N_n$ rules as follows:

$$k(x) = \frac{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_n=1}^{N_n} \bar{y}^{i_1 i_2 \ldots i_n} [\text{«}_{A_1^{i_1}}(x_1) \text{«}_{A_2^{i_2}}(x_2) \cdots \text{«}_{A_n^{i_n}}(x_n)]}{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_n=1}^{N_n} [\text{«}_{A_1^{i_1}}(x_1) \text{«}_{A_2^{i_2}}(x_2) \cdots \text{«}_{A_n^{i_n}}(x_n)]}. \tag{13}$$

(see Lemma 1).

In other words for every 3 points of e^i in $[\sim_i, \ _i]$, relation of membership function can be hold, so in fuzzy sets A^i approximator $k(x)$ is constructed. To define the approximation accuracy of the fuzzy systems, the following theorem has been defined:

Theorem 1 (see [20]). *Let k be the fuzzy system in (13) and g be the unknown function in (12). Let n=2, if G is continuously differentiable on* $U = [\sim_1, \ _1] \times [\sim_2, \ _2]$ *then*

$$\|g - k\|_\infty \le \left\| \frac{\partial g}{\partial x_1} \right\|_\infty h_1 + \left\| \frac{\partial g}{\partial x_2} \right\|_\infty h_2 \tag{14}$$

where

$$h_i = \max_{1 \le j \le N_{i-1}} |e_i^{j+1} - e_i^j| \quad for \quad i = 1, 2 \, .$$

Proof. See [20].

From (13) and the definition of h_i for $i = 1, 2$ we conclude that more accurate approximation can be obtained by defining more fuzzy sets for each x_i

Lemma 3. *Let k(x) be the fuzzy system (13) and* $e_1^{i_1}, e_2^{i_2}$ *be the points defined in the design procedure for k(x) then,*

$$f(e_1^{i_1}, e_2^{i_2}) = g(e_1^{i_1}, e_2^{i_2}) \tag{15}$$

for $i_1 = 1, 2, \ldots, N_1, i_2 = 1, 2, \ldots, N_2$.

Theorem 2. *Let G be a function that is drived from Newton's methodwhich applied for F function in* $U \subseteq \mathbb{R}^n$ *and K be the fuzzy system in (13). If G is continuously differentiable on* $U = [\sim_1, \ _1] \times [\sim_2, \ _2] \times \ldots \times [\sim_n, \ _n]$ *where* $G = [g_1, g_2, \ldots, g_n]^T$ *and* $K = [k_1, k_2, \ldots, k_n]^T$. *Then,*

$$\|g_\neg - k_\neg\|_\infty \le \sum_{i=1}^{n} \left\| \frac{\partial g_\neg}{\partial x_i} \right\|_\infty h_i \quad for \quad \neg = 1, 2, \ldots, n \tag{16}$$

where

$$h_i = \max_{1 \le j \le N_{i-1}} |e_i^{j+1} - e_i^j| \quad for \quad i = 1, 2, \ldots, n$$

Proof. It is straightforward from Theorem 1.

Corollary 1. *As a result of Theorem 2, to obtaine sufficient accuracy by approximator function K. It is sufficient that the vector H satisfies in the following inequality system (17)*:

$$\begin{cases} JH \le V \\ H > 0 \end{cases} \tag{17}$$

where $H = [h_i]$, $J = \left[\left\| \frac{\partial g_{\neg}}{\partial x_i} \right\|_{\infty} \right]$ *and* $V = [\pounds_1, \pounds_2, \dots, \pounds_n]^T$ *for* $i, \neg = 1, 2, \dots, n$ *, and* $\pounds_i$ *is required accuracy.*

We assert the feasible region is non-empty for the above system (17).

Lemma 4. *For a tolerance vector* $v = (\pounds_1, \pounds_2, \dots, \pounds_n)^T$ *the set of all possible vectors H, i. e.*

$$A = \left\{ H \in \mathbb{R}^n : \quad JH \le V, H > 0 \right\} \tag{18}$$

is nonempty.

Proof. Let $\pounds_{\min} = \min_{\neg = 1,2,\dots,n} \{\pounds_{\neg}\}$ and $m = \max_{\neg, i=1,2,\dots,n} J_{\neg i}$. Let

$$\tilde{H} = \frac{1}{mn} [\pounds_{\min}, \dots, \pounds_{\min}]^T ,$$

then we show that $\tilde{H}$ is a feasible point. we have for *v*th row of *JH*:

$$\left(J\tilde{H} \right)_{\neg} = \sum_{i=1}^{n} J_{\neg i} (\tilde{H})_i = \frac{\pounds_{\min}}{mn} \sum_{i=1}^{n} J_{\neg i} \le \frac{\pounds_{\min}}{mn} \sum_{i=1}^{n} m = \pounds_{\min} \le \pounds_{\neg}$$

so $\tilde{H} \in A$. Therefore, the feasible region is not empty.

Consider the system $JH = V$, we approach to obtain the solution of this system such as H' which $H' > 0$, then we put $\hat{H} = H'$ and take H' as an acceptable piont, otherwise if there is no solution or the system has a solution in which for some *i*, $h_i \le 0$. If $\hat{H}$ be choosen from feasible region it is valid for every $0 < H \le \hat{H}$.

For *n*=2 we have the following system of inequalities:

$$\begin{cases} \left\| \frac{\partial g_1}{\partial x_1} \right\|_{\infty} h_1 + \left\| \frac{\partial g_1}{\partial x_2} \right\|_{\infty} h_2 \le \pounds_1 \\ \left\| \frac{\partial g_2}{\partial x_1} \right\|_{\infty} h_1 + \left\| \frac{\partial g_2}{\partial x_2} \right\|_{\infty} h_2 \le \pounds_2 \\ h_1, h_2 > 0 \end{cases} \tag{19}$$

Fig. 2 (a), shows the acceptable point $\hat{H} = H'$ which H' is the solution of the system and $H' > 0$. Otherwise, we have for some $h_i \le 0,\ i = 1, 2.$ With respect to the inequalities in system (19), by defining feasible area every point in feasible region can be chosen as acceptable $\hat{H}$. That is shown in Fig. 2 (b).

4. Examples.

Example 4.1. Consider the following nonlinear system:

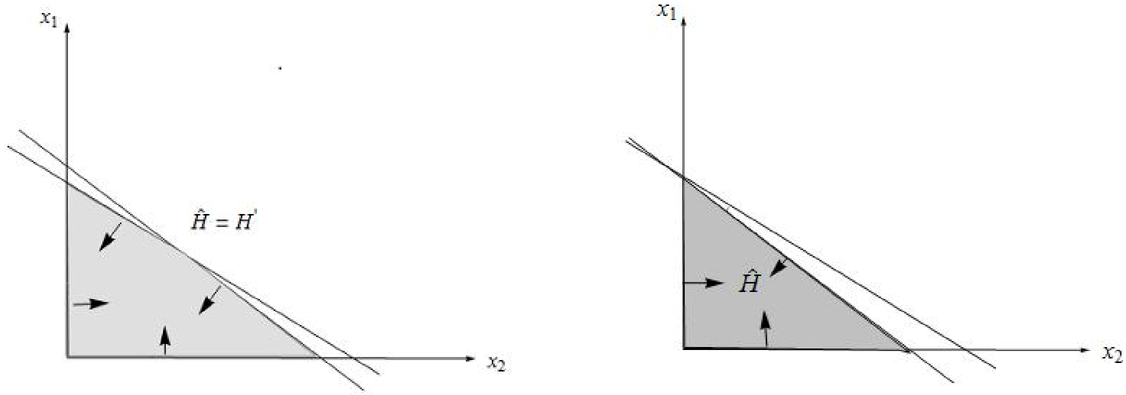

Fig. 2. Illusteraion for the solution; (a) when $H' > 0$, (b) some $h_i < 0$.

$$\begin{cases} x_1 + 2x_2 = 3 \\ 2x_1^2 + x_2^2 = 5 \end{cases}$$

with the solution $(x, y) = (1.488033871712585, 0.7559830641437075)$. Let $f_1(x_1, x_2) = x_1 + 2x_2 - 3$ and $f_2(x_1, x_2) = 2x_1^2 + x_2^2 - 5$. Then the vectorial function by unknown vector $X = [x_1, x_2]$ would be $F(X) = [f_1(X), f_2(X)]$. From recursive Newton fuormulation we have

$$X_n = X_{n-1} - [JF(X)]^{-1} F(X) \tag{20}$$

By designing a fuzzy system, we would approximate Equation (20) on $U = [1.3,1.7] \times [0.5, 0.9]$. Suitable H corresponding to each intervals is obtained from Theorem 2. Here we have $h_1 = 0.01$ and $h_2 = 0.04$ with a required accuracy $£ = 0.1$. There are 41 and 11 fuzzy sets A_i^j on $U = [1.3,1.7] \times [0.5, 0.9]$ respevtively, where $i = 1, 2$ and $j = 1, 2, \ldots, N_i$. The fuzzy system is constructed from 451 rules. The final fuzzy system that approximates the function F, is derived from above system of nonlinear equation (13) as the following form;

$$k(x) = \frac{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \bar{y}^{i_1 i_2} [« _{A_1^{i_1}}(x_1) « _{A_2^{i_2}}(x_2)]}{\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} ł[« _{A_1^{i_1}}(x_1) « _{A_2^{i_2}}(x_2)]}$$

where $N_1 = 41$ and $N_2 = 11$. Since $« _{A_l^i}(x_l) = « _{A_l^i}(x_l; e_l^{j_l - 1}, e_l^{j_l}, e_l^{j_l + 1})$ for $l = 1, 2$. The accuracy of k as an approximator to g in the points of $e_1^{j_1} = 1.3 + 0.01(j_1 - 1)$ for $j_1 = 2, \ldots, 40$ and $e_2^{j_2} = 0.5 + 0.04(j_2 - 1)$ for $j_2 = 2, \ldots, 11$, in comparison to the exact solution is shown in Fig. 3.

We have the best solution where there is the least value of error. For this example $(i, j) = (18, 27)$ with error $= 5.8211e - 5$. It is shown in Table 1 .

Example 4.2. Consider the following nonlinear system

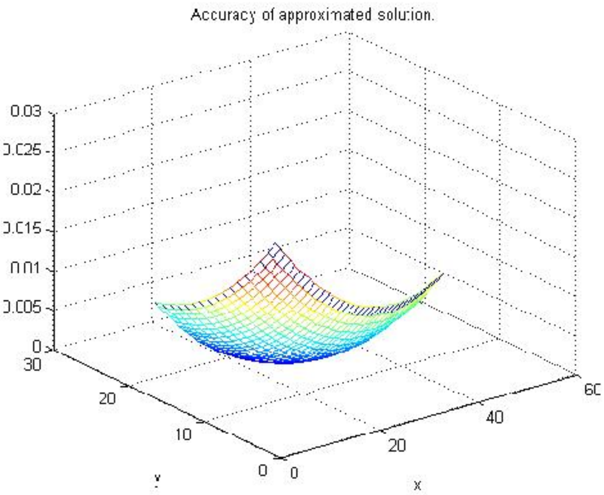

Fig. 3. Comparison of the surface generated from approximator $k(x)$ with the exact solution in Example 1.

$$\begin{cases} x^2 - y = -1 \\ x - \cos(\frac{\pi y}{2}) = 0 \end{cases}$$

with the solution $(x, y) = (-1, 2)$. The bounded set is $U = [-1.2, -0.2] \times [1.8, 2.2]$. In this example we have $h < 0$ so we take h_1 and h_2 from the feasible region such as $h_1 = 0.23$ and $h_2 = 0.09$ with a required accuracy $\varepsilon = 0.1$. The fuzzy system is constructed from 25 rules. As it is mantioned in Example 1, for approximator $k(x)$ we have $N_1 = 5$ and $N_2 = 5$ .The difference between the exact solution and approximator $k(x)$ is shown in Fig. 4.

The least value of error for this example happens in $(i, j) = (39,11)$ with error $1.1211e-5$. Table 2, presents some approximated solution of Example 2.

Example 4.3. Consider the following nonlinear system

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ 2x^2 + y^2 - 4z = 0 \\ 3x^2 - 4y^2 + z^2 = 0 \end{cases}$$

whith the solution $(x, y) = (0.6982886099715139, 0.6285242979602138, 0.3425641896895695)$. The

Table 1. Approximated solutions for Example 1 for *j*=27.

$i = 18$			
j	x_1	x_2	E
25	1.48821	0.75589	$1.9705e-4$
26	1.48810	0.75594	$8.4177e-5$
27	1.48808	0.75595	$5.8211e-5$
28	1.48814	0.75592	$1.1966e-4$
29	1.48827	0.75586	$2.6904e-4$

Table 2. Approximated solutions for Example 2 for *j*=1.

$i = 39$			
j	x_1	x_2	E
9	-0.99855	1.99704	$3.2856e-3$
10	-0.99918	1.99834	$1.8484e-3$
11	-0.99999	1.99998	$1.1211e-5$
12	-0.99994	1.99989	$1.1741e-4$
13	-0.99990	1.99980	$2.1348e-4$

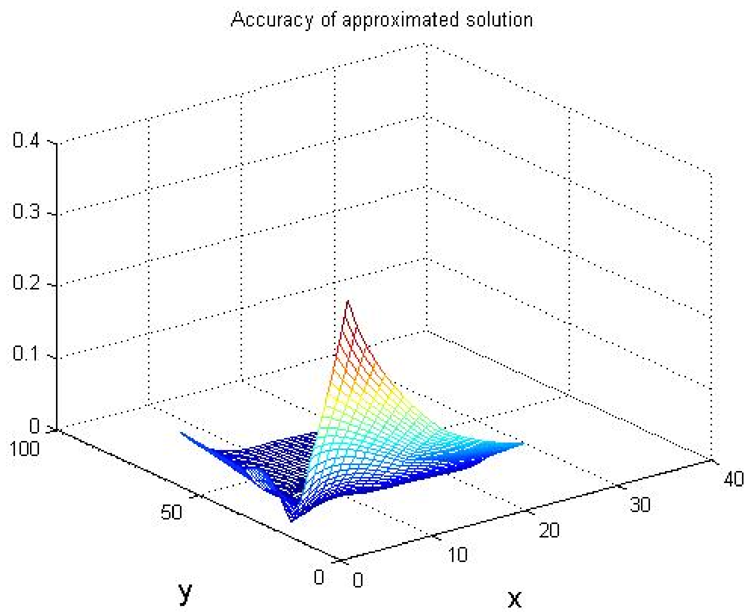

Fig. 4. Comparison of the surface generated from approximator $k(x)$ with the exact solution in Example 2.

bounded set is $U=[0,1]\times[0,1]\times[0,1]$. Here we have $h_1=0.04$, $h_2=0.07$ and $h_3=0.07$ with a required accuracy $\pounds=0.1$. As it is mantioned before for approximator $k(x)$ we have $N_1=25$, $N_2=15$ and $N_3=15$. The acceptable approximation of the solutions is in $(i,j,k)=(25,23,15)$ with error $2.902107e-5$. However Table 3, presents approximated solution

5. Summary and conclusions

Solving nonlinear system of equetions by using approximation of fuzzy systems is presented in this paper. The system is been considered as a function $F(x)=0$. Problem furmolation of the itterative methods such as Newton's or etc. can be transformed to function *G* by operator *P*. To obtain the acceptable approximated solution of nonlinear system, a fuzzy system is designed as an approximator of Newton furmolation, in this paper. To achive our purpose, we used fuzzy system with singleton fuzzifier and center average defuzzifier. Therefore, with respect to the required accuracy, bound of the problem divided by h which is explained in

Table 3. Approximated solutions for Example 3.

k=15 j=23			k=15 j=20			j=23 i=20		
i	x_1	$E(x_1)$	j	x_2	$E(x_2)$	k	x_3	$E(x_3)$
18	0.69862	$3.3871e-4$	21	0.62870	$1.7924e-3$	13	0.34239	$1.0e-3$
19	0.69853	$2.4861e-4$	22	0.62880	$2.8246e-3$	14	0.34241	$1.0e-3$
20	0.69826	$2.0045e-5$	23	0.62850	$2.0872e-4$	15	0.34256	$1.0e-5$
21	0.69800	$2.8102e-4$	24	0.62820	$3.1456e-3$	16	0.34254	$1.0e-4$
22	0.69775	$5.3465e-4$	25	0.62792	$5.9910e-3$	17	0.34253	$1.0e-4$

Theorem 2. Finally, desiged fuzzy system, using membership functions of fuzzy sets A^i approximate $G(x)$, that is derived by Newton furmolation of $F(X)$ with high accuracy. The effectiveness of fuzzy system was demonstrated by computer and using matlab on some examples. Obtained results of solving problems represent the advantage of applying fuzzy system to solve nonlinear systems with comparison by itterative methods such as shorter computation time and approximate solutions in good accuracy. So far, some numerical methods have been investigated by researchers [21-23] among the rest, stochastic global optimization [24], particle swarm optimization (PSO) [25] are remarkable. Since, it's not always possible to obtain exact solution by numerical methods such as Newton-like gradient-based methods, aspect of the impossibility to evaluate derivatives, optimization algorithms can be helpful. Although, these optimization based methods like PSO or stochastic global optimization are capable to obtain solution of system of equations with high accuracy, but, they are inevitable to apply high itteration and increase the computational complexity and also there are highly dependent on stochastic processes and are sensitive to pbest and gbest.The method that is studied in this paper has two outstanding advantages. First, we do not have to compute $JF(X)$, since, operator *k* would approximate Newton's formula and second, avoiding from iteration, it causes less computational complexity.

However, this method is so sensitive to h and because of increasing cost of the computations, it is not suitable to apply for the systems of nonlinear equation with large scales. This paper is one of the first attempts to approximate the solution of nonlinear system of equations by designed fuzzy system and investigation of the method to the case of different itterative furmolation left for future studies.

მათემატიკა

ფაზი-სისტემების გამოყენება არაწრფივ განტოლებათა სისტემის მიახლოებითი ამოხსნებისთვის

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წინამდებარე ნაშრომში განხილულია არაწრფივ განტოლებათა სისტემების ამოხსნა. მიახლოებითი ამოხსნისთვის ჩვენთვის მნიშვნელოვანია ფაზი-სისტემისა და იტერაციული მეთოდების გაერთიანება. წარმოდგენილია შექმნილი ფაზი-სისტემის ელემენტარული თავისებურებანი. ფაზი-სისტემის შესაქმნელად ჩვენ გამოვიყენეთ სინგლტონის (ერთელემენტიანი) ფაზიფიკატორი და ცენტრის საშუალო დეფაზიფიკატორი (დეფაზიფაიერი). შემოთავაზებული მეთოდი ჩვენ გვეხმარება იმ სირთულეების დაძლევაში, რომლებიც წარმოიშობა არაწრფივ განტოლებათა რთული სისტემის

გამოთვლაში სხვადასხვა იტერაციული მეთოდების გამოყენებით. აღნიშნული მეთოდის სისწორის თვალსაჩინოებისათვის, ჩვენ ვიყენებთ მას რამდენიმე მაგალითში არაწრფივ განტოლებათა სისტემის ზოგიერთი ტიპის ჩათვლით.

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