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Physics

Exact Solutions of the Static Cylindrically Symmetric Metric Based on Lyra Geometry for the Viscous Fluid with Massless Scalar Field

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ABSTRACT. We have discussed all the possible solutions of the field equations for the static cylindrically symmetric metric in Lyra geometry in the presence of viscous fluid coupled with a massless scalar field. *© 2016 Bull. Georg. Natl. Acad. Sci.*

Key words: static cylindrically symmetric spacetime, Lyra manifold, viscous fluid.

1. Introduction

Einstein provided a general theory of gravitation by geometry and his theory has been very successful in describing the gravitational phenomena. Einstein field equations without the cosmological constant admitted only nonstatic solutions and he introduced the cosmological constant in order to obtain the static models. The properties of the spacetime require the Riemannian geometry for their description. Several modifications of Riemannian geometry are suggested to unify gravitation, electromagnetism and other effects in universe. One of the modified theories was introduced by Lyra [1]. He introduced an additional gauge function into structureless manifold as a result of which a displacement vector field arises naturally from the geometry. The Einstein field equations in normal gauge based on Lyra were manifold defined by Sen [2] and Sen and Dunn [3] as

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R+\frac{3}{2}\xi_{\alpha}\xi_{\beta}-\frac{3}{4}g_{\alpha\beta}\xi_{\mu}\xi^{\mu}=T_{\alpha\beta}, \tag{1}$$

where ξ_{α} is the Lyra displacement vector field and other symbols have their usual meaning as in Riemannian geometry. We choose the geometric units in which $8\pi G=c=1$. In Lyra formalism, the constant displacement vector field plays the same role as the cosmological constant in the standard general relativity, [4]. Also, the scalar-tensor treatment based on Lyra manifold predicts some effects, within the observational limit, as in Einstein theory [4].

2. The Metric and Field Equations

We assume that the metric of the spacetime, in which the cosmic fluid resides, is of the static cylindrically symmetric form with the following line element, [5]:

$$ds^2 = \lambda(dt^2 - d\rho^2 - \rho^2 d\varphi^2) - \frac{1}{\lambda}dz^2, \tag{2}$$

where λ is unknown function of ρ only. Also, we consider a cosmic fluid endowed with a bulk viscosity η and a shear viscosity δ. The energy-momentum tensor due to this viscous fluid together with a massless scalar field is given by

$$T_{\mu\nu} = \rho_m u_\mu u_\nu - (p - \eta\theta)h_{\mu\nu} - 2\delta\sigma_{\mu\nu} + \Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\sigma}\Phi^{,\sigma}, \tag{3}$$

here ρ_m, p and u_μ are, respectively, the matter density, isotropic pressure and 4-velocity vector of the matter distribution such that $u_\mu = (1,0,0,0)$. The comma denotes partial derivatives with respect to the appropriate coordinates. Also, $h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ is the projection tensor, $\theta = \theta^\mu_\mu = u^\mu_{;\mu}$ is the scalar expansion, the semicolon denotes the covariant differentiation, $\theta_{\mu\nu} = \frac{1}{2}(u_{\mu;\alpha}h^\alpha_\nu + u_{\nu;\alpha}h^\alpha_\mu)$ is the expansion tensor, $\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3}h_{\mu\nu}\theta$ is the shear tensor and Φ is the massless scalar field such that satisfies the Klein-Gordon equation as

$$\Box\Phi = 0, \tag{4}$$

where $\Box\Phi = \frac{1}{\sqrt{-g}}[\sqrt{-g}g^{\mu\nu}\Phi_{,\nu}]_{,\mu}$ while $g = \det(g_{\mu\nu})$. By considering $\Phi = \Phi(\rho,t)$, the equation (4) is changed to the following relation

$$\Phi'' + \frac{1}{\rho}\Phi' - \ddot{\Phi} = 0, \tag{5}$$

where the over head dot and prime indicate partial differentiation with respect to t and ρ respectively. To continue our analysis, we consider the Lyra displacement vector to be a time-like vector as

$$\xi_\mu = (\beta,0,0,0), \tag{6}$$

where β is either a constant or a function. Before proceeding, with a simple calculation we find that $\theta = 0$. Hence, we cannot determine the bulk viscosity for this model. In the next step, the field equations (1) for the spacetime metric (2) lead to the following equations

$$(\dot{\Phi})^2 + (\Phi')^2 + 2\rho_m + 2p(1-\lambda) - \frac{3}{2}\beta^2 - \frac{a^2}{2} = 0, \tag{7}$$

$$(\dot{\Phi})^2 + (\Phi')^2 + 2\lambda p - \frac{3}{2}\beta^2 - \frac{a^2}{2} = 0, \tag{8}$$

$$(\dot{\Phi})^2 - (\Phi')^2 + 2\lambda p - \frac{3}{2}\beta^2 + \frac{a^2}{2} = 0, \tag{9}$$

$$(\dot{\Phi})^2-(\Phi')^2+2\lambda p-\frac{3}{2}\beta^2+2a'+\frac{a^2}{2}+\frac{2a}{\rho}=0, \tag{10}$$

$$2\lambda\dot{\Phi}\Phi'+a\delta(2\lambda-1)=0, \tag{11}$$

where $a=\frac{\lambda'}{\lambda}$ and the quantities p, ρ_m and δ depend on ρ only. By comparing the equations (9) and (10), we get

$$a'+\frac{1}{\rho}a=0. \tag{12}$$

By solving this equation, we conclude

$$\lambda=\lambda_0\rho^n, \tag{13}$$

where n and λ_0 are constants. From the equations (8) and (9) we find that

$$\Phi=\pm\frac{n}{\sqrt{2}}\ln\rho+\psi(t), \tag{14}$$

such that ψ is an arbitrary function. This function can be obtained from the equation (5) as

$$\psi=ct+b, \tag{15}$$

where b and c are constants. Therefore, we have

$$\Phi=\pm\frac{n}{\sqrt{2}}\ln\rho+ct+b. \tag{16}$$

Next, with the help of the equations (11) and (16), we deduce

$$\delta=\pm\frac{c\lambda\sqrt{2}}{1-2\lambda}. \tag{17}$$

The shear viscosity has singularity at $\lambda=\frac{1}{2}$. Also, using the equations (7) and (8), become

$$p=\frac{\omega}{\lambda}, \tag{18}$$

$$\rho_m=(2-\frac{1}{\lambda})\omega, \tag{19}$$

where $\omega=\frac{3}{4}\beta^2-\frac{c^2}{2}$. From the reality conditions, i.e. $p>0$ and $\rho_m>0$, we conclude

$$\lambda>\frac{1}{2}. \tag{20}$$

This means that $\omega>0$ which is equivalent to

$$|\beta|>\sqrt{\frac{2}{3}}|c|. \tag{21}$$

Consequently, by using the condition (20), we lead to

$$\delta=\frac{\lambda|c|\sqrt{2}}{2\lambda-1}. \tag{22}$$

3. Physical Models

We assume that the fluid obeys an equation of state of the following form

$$p = (\gamma - 1)\rho_m \;\; ; \;\; 1 \le \gamma \le 2, \tag{23}$$

where γ is a constant. Below we will discuss the physical models corresponding to $\gamma = 1, \frac{4}{3}, 2$:

Case 1. Dust distribution model (γ =1)

From equation (18), we have $p = \omega = 0$ and so $\rho_m = 0$. Hence, we can see that this case is not possible for metric (2).

Case 2. Radiating model ($\gamma = \frac{4}{3}$)

The physical quantities in this case take the following forms

$$p = \frac{1}{3}\rho_m = \frac{\omega}{2} \;\; , \;\; \delta = \frac{2\sqrt{2}}{3} | c |, \tag{24}$$

here $\lambda = 2$. We observe that the metric (2) becomes a flat spacetime.

Case 3. Zel'dovich fluid model (γ =2)

In this case, we find that

$$p = \rho_m = \omega \;\; , \;\; \delta = \sqrt{2} | c |, \tag{25}$$

where $\lambda = 1$. Consequently, the spacetime must be a flat spacetime.

ფიზიკა

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