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Mathematics

Relationship between Homology of a Simplicial Semimodule and Homology of its Module Completion

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ABSTRACT. Let $K(\Lambda)$ denote the ring completion of a semiring Λ and let S be a simplicial Λ-semimodule, $H_n(S)$ the *n*-th homology Λ-semimodule of S introduced in our earlier paper, $K(S)$ the $K(\Lambda)$-module completion of S, $H_n(K(S))$ the *n*-th homology $K(\Lambda)$-module of $K(S)$ and $k_S : S \to K(S)$ the canonical simplicial map. We prove (1) that the induced map $H_n(k_S) : H_n(S) \to H_n(K(S))$ is an injective Λ-homomorphism for all *n*; (2) that if S satisfies the Kan condition and the Λ-semimodule of path components of S is a $K(\Lambda)$-module, then $H_n(S)$ is a $K(\Lambda)$-module and the induced map $H_n(k_S) : H_n(S) \to H_n(K(S))$ is a $K(\Lambda)$-isomorphism for all *n*. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Key words: semimodule, chain complex of semimodules, simplicial semimodule, homology semimodule, module completion

Let Λ be a semiring and let $K(\Lambda)$ be its ring completion. In [1], we introduced and studied homology Λ-semimodules $H_n(S)$ of a presimplicial -semimodule S and indicated some applications of them. The purpose of the present paper is to examine relationship between $H_n(S)$ and $H_n(K(S))$, where S is a simplicial Λ-semimodule, $K(S)$ is its $K(\Lambda)$-module completion, $H_n(S)$ the *n*-th homology Λ-semimodule of S and $H_n(K(S))$ the *n*-th homology $K(\Lambda)$-module of $K(S)$.

By a semiring Λ we mean an algebraic structure $(\Lambda,+,0,\cdot,1)$ in which $(\Lambda,+,0)$ is an abelian monoid, $(\Lambda,\cdot,1)$ a monoid, and

$$\alpha\cdot(\alpha'+\alpha'')=\alpha\cdot\alpha'+\alpha\cdot\alpha'',$$
$$(\alpha'+\alpha'')\cdot\alpha=\alpha'\cdot\alpha+\alpha''\cdot\alpha,$$
$$\alpha\cdot 0=0\cdot\alpha=0$$

for all $\alpha,\alpha',\alpha''\in\Lambda$.

Let Λ be a semiring. An abelian monoid $A=(A,+,0)$ together with a map $\Lambda\times A\to A$, written as

$(\alpha, a) \mapsto \alpha a$, is called a (left) Λ-semimodule if

$$\alpha(a + a') = \alpha a + \alpha a',$$
$$(\alpha + \alpha')a = \alpha a + \alpha' a,$$
$$(\alpha \cdot \alpha')a = \alpha(\alpha' a),$$
$$1a = a, \quad 0a = 0$$

for all $\alpha, \alpha' \in \Lambda$ and $a, a' \in A$. It immediately follows that $\alpha 0 = 0$ for any $\alpha \in \Lambda$.

A Λ-homomorphism $f : A \to B$ between Λ-semimodules *A* and *B* is defined in the standard manner.

Note that $\mathbb{N}$ -semimodules, where $\mathbb{N}$ is the semiring of nonnegative integers, are precisely abelian monoids.

Recall that the group completion of an abelian monoid *M* can be constructed in the following way. Define an equivalence relation ~ on $M \times M$ as follows:

$$(u, v) \sim (x, y) \Leftrightarrow u + y + z = v + x + z \quad \text{for some} \quad z \in M.$$

Let [u,v] denote the equivalence class of (*u*,*v*). The quotient set ($M \times M$)/~ with the addition $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$ is an abelian group $(0 = [x, x], -[x, y] = [y, x])$. This group, denoted by $K(M)$, is the group completion of *M*, and $k_M : M \to K(M)$ defined by $k_M(x) = [x, 0]$ is the canonical homomorphism. If *M* is a semiring, then the multiplication $[x_1, y_1] \cdot [x_2, y_2] = [x_1 x_2 + y_1 y_2, x_1 y_2 + y_1 x_2]$ converts $K(M)$ into the ring completion of the semiring *M*, and k_M into the canonical semiring homomorphism. Now assume that *A* is a Λ-semimodule. Then $K(A, +, 0)$ with the multiplication

$$[\alpha_1, \alpha_2][a_1, a_2] = [\alpha_1 a_1 + \alpha_2 a_2, \alpha_1 a_2 + \alpha_2 a_1], \quad \alpha_1, \alpha_2 \in \Lambda, \quad a_1, a_2 \in A,$$

becomes a $K(\Lambda)$ -module. This $K(\Lambda)$ -module, denoted by $K(A)$, is the $K(\Lambda)$ -module completion of the Λ-semimodule *A*, and $k_A : A \to K(A), \quad k_A(a) = [a, 0],$ is the canonical Λ-homomorphism.

A Λ-semimodule *A* is said to be cancellative if whenever $a + a' = a + a'', a, a', a'' \in A$, holds, one has $a' = a''$. Obviously, *A* is cancellative if and only if the canonical Λ-homomorphism $k_A : A \to K(A)$ is injective.

A Λ-semimodule *A* is called a Λ-module if $(A, +, 0)$ is an abelian group. One can easily see that *A* is a Λ-module if and only if *A* is a $K(\Lambda)$ -module. Hence, if *A* is a Λ-module, then $K(A) = A$ and $k_A = 1_A$.

For more information about semimodules, see [2].

Definition 1 ([1]). We say that a sequence of Λ-semimodules and Λ-homomorphisms

$$X : \cdots \rightrightarrows X_{n+1} \overset{\partial_{n+1}^+}{\underset{\partial_{n+1}^-}{\rightrightarrows}} X_n \overset{\partial_n^+}{\underset{\partial_n^-}{\rightrightarrows}} X_{n-1} \rightrightarrows \cdots, \qquad n \in \mathbb{Z},$$

written $X = \{X_n, \partial_n^+, \partial_n^-\}$ for short, is a *chain complex* if

$$\partial_n^+ \partial_{n+1}^+ + \partial_n^- \partial_{n+1}^- = \partial_n^+ \partial_{n+1}^- + \partial_n^- \partial_{n+1}^+$$

for each integer *n*. For every chain complex *X*, we define the Λ-semimodule

$$Z_n(X) = \left\{ x \in X_n \mid \partial_n^+(x) = \partial_n^-(x) \right\},$$

the *n-cycles*, and the *n-th homology* Λ*-semimodule*

$$H_n(X) = Z_n(X) / \circ_n(X),$$

where $\circ_n(X)$ is a congruence on the Λ-semimodule $Z_n(X)$ defined as follows:

$$x \circ_n(X)\, y \;\Leftrightarrow\; x + \partial_{n+1}^+(u) + \partial_{n+1}^-(v) = y + \partial_{n+1}^+(v) + \partial_{n+1}^-(u) \text{ for some } u, v \in X_{n+1}.$$

The Λ-homomorphisms $\partial_n^+, \partial_n^-$ are called *differentials* of the chain complex *X*.

A sequence $G = \{G_n, d_n^+, d_n^-\}$ of Λ-modules and Λ-homomorphisms is a chain complex if and only if

$$\cdots \longrightarrow G_n \xrightarrow{d_n^+ - d_n^-} G_{n-1} \longrightarrow \cdots$$

is an ordinary chain complex of Λ-modules. Obviously, for any chain complex $G = \{G_n, d_n^+, d_n^-\}$ of Λ-modules, the homology $H_*(G)$ coincides with the usual homology $H_*(\{G_n, d_n^+ - d_n^-\})$.

Definition 2 ([1]). Let $X = \{X_n, \partial_n^+, \partial_n^-\}$ and $X' = \{X'_n, \partial_n'^+, \partial_n'^-\}$ be chain complexes of Λ-semimodules. We say that a sequence $f=\{f_n\}$ of Λ-homomorphisms $f_n : X_n \to X'_n$ is a $\pm$-*morphism* from *X* to X' if

$$f_{n-1}\partial_n^+ = \partial_n'^+ f_n \quad \text{and} \quad f_{n-1}\partial_n^- = \partial_n'^- f_n \quad \text{for all } n.$$

If $f=\{f_n\} : X \to X'$ is a $\pm$-morphism of chain complexes, then $f_n(Z_n(X)) \subset Z_n(X')$, and the map

$$H_n(f) : H_n(X) \to H_n(X'), \quad H_n(f)(cl(x)) = cl(f_n(x)),$$

is a Λ-homomorphism. Thus H_n is a covariant additive functor from the category of chain complexes and their $\pm$-morphisms to the category of Λ-semimodules.

For any chain complex $X = \{X_n, \partial_n^+, \partial_n^-\}$ of Λ-semimodules,

$$K(X) : \cdots \longrightarrow K(X_{n+1}) \xrightarrow{K(\partial_{n+1}^+) - K(\partial_{n+1}^-)} K(X_n) \xrightarrow{K(\partial_n^+) - K(\partial_n^-)} K(X_{n-1}) \longrightarrow \cdots$$

is an ordinary chain complex of $K(\Lambda)$-modules (that is, Λ-modules). The canonical $\pm$-morphism $k_X = \{k_{X_n} : X_n \to K(X_n)\}$ from the chain complex *X* to the chain complex $\{K(X_n), K(\partial_n^+), K(\partial_n^-)\}$ induces the Λ-homomorphism

$$H_n(k_X) : H_n(X) \to H_n(K(X)), \quad H_n(k_X)(cl(x)) = cl([x, 0]),$$

for all *n*. One can easily check that if *X* is a chain complex of cancellative Λ-semimodules, then $H_n(k_X)$ is an injection.

Definition 3. We say that a chain complex $X = \{X_n, \partial_n^+, \partial_n^-\}$ of Λ-semimodules is regular if whenever

$$x + \partial_{n+1}^+(u) + \partial_{n+1}^-(v) + z = y + \partial_{n+1}^+(v) + \partial_{n+1}^-(u) + z, \quad x, y \in Z_n(X), \quad z \in X_n, \quad u, v \in X_{n+1},$$

holds, we have

$$x + \partial_{n+1}^+(a) + \partial_{n+1}^-(b) = y + \partial_{n+1}^+(b) + \partial_{n+1}^-(a), \quad a, b \in X_{n+1}.$$

It is obvious that any chain complex of cancellative Λ-semimodules $X = \{X_n, \partial_n^+, \partial_n^-\}$ is regular. For a Λ-semimodule *A*,

$$\cdots \rightrightarrows A \underset{1}{\overset{1}{\rightrightarrows}} A \underset{0}{\overset{\overline{m}}{\rightrightarrows}} A \underset{1}{\overset{1}{\rightrightarrows}} A \underset{0}{\overset{\overline{m}}{\rightrightarrows}} A \rightrightarrows \cdots, \quad m \in \mathbb{N}, \quad m \geq 1, \quad \overline{m}(a) = ma,$$

is an example of a regular chain complex of (not necessarily cancellative) Λ-semimodules.

Proposition 4. *Let* $X = \{X_n, \partial_n^+, \partial_n^-\}$ *be a chain complex of* Λ*-semimodules. The induced* Λ*-homomorphism*

$$H_n(k_X) : H_n(X) \to H_n(K(X)), \quad H_n(k_X)(cl(x)) = cl([x, 0]),$$

is an injection for all n if and only if X is regular.

Recall that a presimplicial Λ-semimodule *S* is a sequence of Λ-semimodules $S_0, S_1, S_2, \ldots$ together with Λ-homomorphisms, called face operators,

$$\partial_n^i : S_n \to S_{n-1}, \quad n \geq 1, \quad 0 \leq i \leq n,$$

such that

$$\partial_n^i \partial_{n+1}^j = \partial_n^{j-1} \partial_{n+1}^i \quad \text{if} \quad 0 \leq i < j \leq n+1.$$

Suppose $S = \{S_n, \partial_n^i\}$ and $T = \{T_n, \text{¢}_n^i\}$ are presimplicial Λ-semimodules. A morphism (or a presimplicial map) $f : S \to T$ is a collection of Λ-homomorphisms $f_n : S_n \to T_n$ satisfying $f_{n-1}\partial_n^i = \text{¢}_n^i f_n$ for all *i* and for all *n*.

If *S* is a presimplicial Λ-semimodule, then

$$\underline{S} : \cdots \rightrightarrows S_n \overset{\partial_n^+}{\underset{\partial_n^-}{\rightrightarrows}} S_{n-1} \rightrightarrows \cdots \rightrightarrows S_2 \overset{\partial_2^+}{\underset{\partial_2^-}{\rightrightarrows}} S_1 \overset{\partial_1^+}{\underset{\partial_1^-}{\rightrightarrows}} S_0 \rightrightarrows 0,$$

where

$$\partial_n^+ = \partial_n^0 + \partial_n^2 + \cdots, \qquad \partial_n^- = \partial_n^1 + \partial_n^3 + \cdots,$$

is a nonnegative chain complex of Λ-semimodules, called the *standard chain complex* associated to *S* [1]. Using the greatest integer function, one can write

$$\partial_n^+ = \sum_{k=0}^{[\frac{n}{2}]} \partial_n^{2k}, \quad \partial_n^- = \sum_{k=0}^{[\frac{n-1}{2}]} \partial_n^{2k+1}.$$

We define the *n-th homology* Λ*-semimodule of the presimplicial* Λ*-semimodule S* by

$$H_n(S) = H_n(\underline{S}).$$

Clearly, if $f = \{f_n\}$ is a morphism from a presimplicial Λ-semimodule $S = \{S_n, \partial_n^i\}$ to a presimplicial Λ-semimodule $T = \{T_n, \text{¢}_n^i\}$, then $\text{¢}_n^+ f_n = f_{n-1}\partial_n^+$ and $\text{¢}_n^- f_n = f_{n-1}\partial_n^-$ for all $n \geq 1$, that is, *f* can be regarded as a ±-morphism from $\underline{S}$ to $\underline{T}$. Consequently, $H_n(S)$ is a covariant additive functor from the category of presimplicial Λ-semimodules and their morphisms to the category of Λ-semimodules.

Next recall that a simplicial Λ-semimodule is a presimplicial Λ-semimodule *S* together with degeneracy Λ-homomorphisms

$$s_n^i : S_n \to S_{n+1}, \quad 0 \leq i \leq n,$$

satisfying

$$\partial_{n+1}^i s_n^j = \begin{cases} s_{n-1}^{j-1}\partial_n^i, & i < j, \\ \text{id}, & i = j, \ j+1, \\ s_{n-1}^j \partial_n^{i-1}, & i > j+1, \end{cases}$$

and

$$s_{n+1}^i s_n^j = s_{n+1}^{j+1} s_n^i, \qquad i \leq j.$$

Elements of S_n are called *n*-simplices.

Let *S* and S' be simplicial Λ-semimodules. A simplicial map $f : S \to S'$ is a family of Λ-homomorphisms $(f_n : S_n \to S'_n)_{n \geq 0}$ which commute with the face and degeneracy operators.

Proposition 5. *For any simplicial* Λ*-semimodule* $S = \{S_n, \partial_n^i\}$*, the standard chain complex*

$$\underline{S}:\cdots \rightrightarrows S_n \underset{\partial_n^-}{\overset{\partial_n^+}{\rightrightarrows}} S_{n-1} \rightrightarrows \cdots \rightrightarrows S_2 \underset{\partial_2^-}{\overset{\partial_2^+}{\rightrightarrows}} S_1 \underset{\partial_1^-}{\overset{\partial_1^+}{\rightrightarrows}} S_0 \rightrightarrows 0$$

associated to S is regular.

As a corollary of Propositions 4 and 5, we have

Theorem 6. *Suppose that S is a simplicial* Λ*-semimodule,* $K(S)$ *its* $K(\Lambda)$*-module completion and* $k_S : S \to K(S)$ *the canonical simplicial map. Then the induced* Λ*-homomorphism*

$$H_n(k_S): H_n(S) \to H_n(K(S)), \qquad H_n(k_S)(cl(s)) = cl([s,0]),$$

is an injection for each n.

One says that a simplicial Λ-semimodule *S* satisfies the Kan condition if for every collection of *n*+1 *n*-simplices $x_0, x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+1}$ satisfying the compatibility condition $\partial_n^i(x_j) = \partial_n^{j-1}(x_i)$, $i < j$, $i \neq k$, $j \neq k$, there exists an (*n*+1)-simplex *x* such that $\partial_{n+1}^i(x) = x_i$ for $i \neq k$, (see e.g. [3]).

Theorem 7. *Let S be a simplicial* Λ*-semimodule,* $K(S)$ *its* $K(\Lambda)$*-module completion and* $k_S : S \to K(S)$ *the canonical simplicial map. If S satisfies the Kan condition and the* Λ*-semimodule of path components of S is a* $K(\Lambda)$*-module, then* $H_n(S)$ *is a* $K(\Lambda)$*-module and the induced map*

$$H_n(k_S): H_n(S) \to H_n(K(S)), \qquad H_n(k_S)(cl(s)) = cl([s,0]),$$

is a $K(\Lambda)$*-isomorphism for all n.*

As noted above, semimodules over the semiring of nonnegative integers are precisely abelian monoids. Hence, we have the following corollaries.

Corollary 8. *Suppose that A is a simplicial abelian monoid,* $K(A)$ *its group completion and* $k_A : A \to K(A)$ *the canonical simplicial map. Then the induced homomorphism*

$$H_n(k_A): H_n(A) \to H_n(K(A)), \qquad H_n(k_A)(cl(a)) = cl([a,0]),$$

is an injection for each n.

Corollary 9. *Let A be a simplicial abelian monoid,* $K(A)$ *its group completion and* $k_A : A \to K(A)$ *the canonical simplicial map. If A satisfies the Kan condition and the monoid of path components of A is a group, then* $H_n(A)$ *is a group and the induced map*

$$H_n(k_A): H_n(A) \to H_n(K(A)), \qquad H_n(k_A)(cl(a)) = cl([a,0]),$$

is an isomorphism for all n.

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მათემატიკა

კავშირი სიმპლიციალური ნახევრადმოდულის და მისი სიმპლიციალურ მოდულამდე გასრულების ჰომოლოგიებს შორის

ა. პაჭკორია

ივანე ჯავახიშვილის სახელობის თბილისის სახელმწიფო უნივერსიტეტი, ა. რაზმაძის მათემატიკის ინსტიტუტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ხ. ინასარიძის მიერ)

ვთქვათ K(i) აღნიშნავს i ნახევრადრგოლის რგოლამდე გასრულებას და ვთქვათ S არის სიმპლიციალური i -ნახევრადმოდული, $H_n(S)$ - მისი n-ური ჰომოლოგიის i -ნახევრადმოდული, K(S) - S-ის სიმპლიციალურ K(i)-მოდულამდე გასრულება, $H_n(K(S))$ - K(S)-ის n-ური ჰომოლოგიის K(i)-მოდული და K_S: S ì K(S) - კანონიკური ასახვა. ჩვენ ვამტკიცებთ: (1) ინდუცირებული ასახვა $H_n(K_S) : H_n(S)$ ì $H_n(K(S))$ არის ინექციური i -ჰომომორფიზმი ყოველი n-თვის; (2) თუ S აკმაყოფილებს კანის პირობას და S -ის გზების კომპონენტების i -ნახევრადმოდული არის K(i)-მოდული, მაშინ $H_n(S)$ არის K(i)-მოდული და ინდუცირებული $H_n(K_S) : H_n(S)$ ì $H_n(K(S))$ ასახვა არის K(i)-იზომორფიზმი ყოველი n-თვის.

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