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Mathematics

Equilibria of Point Charges on Elastic Contour

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ABSTRACT. We discuss several problems concerned with equilibrium configurations of point charges confined to an elastic piecewise differentiable contour in the plane. The potential energy considered is the sum of Coulomb energy and elastic energy of stretched contour expressed by Hook law. This setting is motivated by the concept of necklace with charged beads investigated by P.Exner. Charges are assumed to be repelling, i.e. they are all of the same sign, but not necessarily equal. We prove that the equilibrium configuration of lowest energy is unique for any values of given charges and has a shape of convex polygon with charges at its vertices. We also describe certain convex polygons which can serve as equilibrium configurations of repelling charges on a given planar Hookean contour. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Key words: point charge, Coulomb energy, Hook energy, equilibrium configuration, planar contour, convex polygon

The input for the problem N(n, q; L, k) considered in this paper consists of an ordered (n+2)-tuple of positive numbers $(q_1, \dots , q_n; L, k)$. The members of n-tuple $q = (q_1, \dots , q_n)$ are interpreted as values of point charges with Coulomb interaction. The rest two numbers indicate that the charges are confined to a Hookean elastic contour (simple closed curve) in the plane with unstretched length L and Hook coefficient (stiffness) k. Such an interpretation is motivated by the concept of necklace with charged beads considered in [1, 2], where the contour was assumed to be flexible but of a fixed length L. The latter model was further investigated in [3, 4] and the present paper is the next step in the same direction. In fact, the setting described below can be considered as simultaneously generalizing Exner's model and polygonal linkages.

The main novelty of our setting is that instead of considering a planar contour of fixed length we assume that it can increase its length due the Coulomb interaction of repelling point charges placed on it. Such a system of constrained charges may be relevant to certain models of mathematical physics. As in [3] we are interested in describing the shapes and energies of possible equilibrium configurations of such a model.

In this paper, for simplicity and clarity we assume that the contour is Hookean, i.e. its elastic energy is proportional to the square of the increment of its length. Correspondingly, the potential energy of a configuration of this system is defined as the sum of Coulomb energy of charges in given positions and elastic energy of the stretched contour. For brevity we refer to it as *Coulomb-Hook energy* (CH-energy). As usual an equilibrium configuration is defined as a critical point of this energy function. The configuration of the lowest possible CH-energy is called the *ground state*.

In the sequel we discuss two problems naturally arising in this context. The first problem, called the *direct electrostatic problem* (DEP), is to find the exact shape and energy of the ground state. Our first main result states that the ground state has a shape of convex polygon with the charges q_j placed at its vertices. So it is sufficient to find the sidelengths and angles of the ground state. The second problem, called the *inverse electrostatic problem* (IEP), is to find out which convex n-gons can be represented as ground states of such a system of n charges on Hookean contour. If such charges exist they are called *stationary charges*. It should be noted that in [3] both these problems have been solved for three point charges on a flexible planar contour of fixed length. The approach of [3] served as a paradigm for the research presented in this paper.

Remark 1. There are several immediate further problems. Do there exist local minima of energy in addition to the global one? Do there exist critical points which are not local minima of energy? Do all of them have a shape of polygon (possibly self-intersecting)? Are the critical points generically non-degenerate and if so, how can one compute their Morse indices? Do the answers depend on the order of charges? What happens if all charges are equal? For a fixed total charge Q, which distribution of charges has the longest ground state?

Our previous study of similar topics suggests several plausible conjectures concerned with the above problems, some of which are given below. In particular, we believe that there may exist several local minima and non-minimal (non-stable) equilibrium configurations of necklace and the full collection of critical points depends on the order of charges. Equilibrium configurations should be generically non-degenerate. For equal charges everything can be calculated explicitly including the Morse indices. The longest ground state is probably achieved if all charges are equal to Q/n.

In the sequel we prove some of these conjectures in several special cases. The general case remains widely open and hopefully deserves attention for the sake of arising non-trivial mathematical aspects. The following basic result is a direct generalization of Proposition 1 in [4] and can be proved in an analogous way.

Theorem 1. *For* $n \geq 3$*, the absolute minimum of CH-energy of Hookean contour with n charges is achieved at a configuration having a shape of convex n-gon with charges at its vertices.*

Remark 2. Notice that we do not claim that all equilibrium configurations have polygonal shapes. This seems highly plausible but we do not have a proof in the general case. If all charges are equal then the ground state has a shape of regular polygon. In this case, we compute below the energy of ground state by a way of analogy with the main result of [1].

Remark 3. In a certain sense, as $k \to \infty$ our model tends to the model of unstretchable string considered by P. Exner. Indeed, as we will see, many entities computed below tend to similar ones given in [1]. In the sequel we will repeatedly use a simple algebraic result which is formulated explicitly for further reference.

Lemma 1. *The roots of cubic equation* $x^3+ax^2+b=0$ *are given by the formula:*

$$x=-\frac{a}{3}+\sqrt[3]{-\frac{a^3}{27}-\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3b}{27}-\frac{25a^6}{19683}}}+\sqrt[3]{-\frac{a^3}{27}-\frac{b}{2}-\sqrt{\frac{b^2}{4}+\frac{a^3b}{27}-\frac{25a^6}{19683}}}. \quad (1)$$

The proof is obvious. One transforms the above equation to a canonical form $y^3 + py + q = 0$ by putting $x = y - a/3$, and uses Cardano-Tartaglia formula for y.

Example. To clarify our setting and aims let us discuss in some detail a simple case of two point charges on a Hookean contour. Consider a Hookean contour X(L, k) with unstretched length L and stiffness k. Let Q > q > 0 be the values of two point charges confined to X(L, k). It is obvious that there is a unique equilibrium configuration represented by a straight line segment I with the two charges at the opposite ends of I. Denote by x the length of I. Obviously, x > L/2. The CH-energy in this case is $E = qQ/x + 2k(x-L)^2$, and the equilibrium equation is $qQ/x^2 = k(x-L)$. This yields a cubic equation for the length x: $x^3 - Lx^2 - qQ/k = 0$. Applying Lemma 1 in this case we get that the roots are given by the formula

$$x=\frac{L}{3}+\sqrt[3]{\left(-\frac{L^3}{27}-\frac{qQ}{2k}+\sqrt{\frac{2L^6}{729}+\frac{qQL}{27k}+\frac{q^2Q^2}{4k^2}}\right)}+ \sqrt[3]{\left(-\frac{L^3}{27}-\frac{qQ}{2k}-\sqrt{\frac{2L^6}{729}+\frac{qQL}{27k}+\frac{q^2Q^2}{4k^2}}\right)}. \quad (2)$$

It is obvious that the discriminant of the above cubic equation is positive. So it has just one real root which is obtained by taking the unique real values of the both cubic roots in the above formula. It is then easy to show that x > L/2 as required. This yields the following result.

Proposition 1. Any two repelling charges on a Hookean loop have a unique equilibrium configuration which is represented by a segment of length x given by formula (2) with the two charges at its ends.

Now one can compute the CH-energy of ground state but the resulting formula is rather ugly and useless. It seems more instructive to consider several limiting cases. It is easy to show that, for fixed charges, the limit of x as $k \to \infty$ equals L/2, which agrees with the answer for the unstretchable flexible loop. One can also fix the total charge S = q + Q and ask for what values of q and Q the ground state configuration of a given Hookean loop has the maximal length. Using the above formulas it is easy to show that the length of equilibrium is maximal when both charges are equal to S/2. At the other extremity, if one of the charges tends to zero the formulas show that the length of ground state tends to L/2. Both these observations agree with the physical intuition and can be extended to many other cases. In the next section we show that, essentially, the same argument yields an explicit answer in the case of three charges.

Let E = E(3; q; L, k) denote the CH-energy considered on the space of configurations of Hookean contour with three point charges. By Theorem 1 the absolute minimum of E has the form of a triangle with charges q_1, q_2, q_3 at its vertices. Let the lengths of its sides be x, y, z. Obviously, x + y + z > L and the potential energy of this configuration can be written in the form

$$E=\frac{q_1q_2}{z}+\frac{q_2q_3}{x}+\frac{q_1q_3}{y}+k(x+y+z-L)^2. \quad (3)$$

The system of equations for the critical point takes the form

$$\frac{q_2q_1}{z^2}=2k(x+y+z-L),\ \frac{q_2q_3}{x^2}=2k(x+y+z-L),\ \frac{q_1q_3}{y^2}=2k(x+y+z-L).$$

Hence the left-hand sides of these three equations are all equal, which enables us to express all sidelengths as multiples of x:

$$\frac{q_1q_3}{y^2}=2k(x+y+z-L). \quad (4)$$

After elementary algebraic manipulations we get a cubic equation for x of the form

$$Ax^3 - Lx^2 - \frac{q_2 q_3}{2k} = 0, \text{ where } A = \frac{\sqrt{q_1 q_2} + \sqrt{q_2 q_3} + \sqrt{q_1 q_3}}{\sqrt{q_2 q_3}}.$$

This equation can again be solved using Lemma 1. The formulas for y and z are obtained by obvious cyclic permutations, which yields an explicit formula for the absolute minimum of potential energy. This implies that, for equal charges, the ground state is represented by a regular triangle as one could expect for symmetry reasons. It is easy to verify that the discriminant of the arising cubic equation is positive so the real solution is unique. Using computer algebra one can now get an explicit formula for the Hessian matrix of CH-energy at the ground state and verify that it is positive definite. Thus the configuration given by these formulas is indeed a stable non-degenerate absolute minimum. We omit the resulting formulas and details of computations since they are pretty standard. For further reference we give a concise formulation of the final conclusion.

Theorem 2. *For any three positive charges, the ground state configuration of the corresponding Hookean contour is unique and its sides can be expressed as explicit functions of* q_1, q_2, q_3, L, k.

Passing to the limit as $k \to \infty$ one can verify that our solution tends to the solution obtained in [3] for unstretchable flexible loop of length L, which shows that the above result gives a consistent generalization of the main result of [3]. In fact, it is possible to prove that the ground state is non-degenerate in the sense of Morse theory. Thus direct problem DEP has an explicit solution for n=2 and n=3. Unfortunately, the situation becomes much more complicated for $n \geq 4$ and most likely the solutions cannot be computed explicitly. However this can be done for certain special collections of point charges. For arbitrary n, we can only solve the case of equal charges using the results for unstretchable loop obtained in [1].

If all *n* charges are equal, say $q_j = q$, then a regular n-gon is a natural candidate for the shape of the ground state. It turns out that this is really so and the perimeter and CH-energy of the ground state can be expressed by explicit formulas.

Theorem 3. *The ground state of n equal charges on a Hookean loop has the shape of regular n-gon the side of which is given by formula (1) with*

$$a = \frac{kL}{n}, \quad b = \frac{q^2}{4n^2} \sin\frac{\pi}{2n}.$$

In fact, this configuration is a non-degenerate local minimum. One can also find the asymptotics of the side and electrostatic energy of this configuration as $n \to \infty$. The proof of this result essentially relies on the calculations presented in [1]. In particular, we use the following formula given in [1].

Proposition 2. The electrostatic energy E(n, q, r) of a system of equal charges q placed at the vertices of regular n-gon inscribed in a circle of radius r is given by the formula

$$E(n,q,r) = \frac{q^2}{4rn} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)}. \tag{5}$$

This in fact follows directly from a well known geometric result which is presented below for convenience of the reader.

Lemma 2. *The length of diagonal joining* j*-th and* i*-th vertices of regular* n*-gon with side* a *is*

$$a\sqrt{2}\sqrt{1 - \cos^2\left(\frac{2\pi(i-j)}{n}\right)}. \tag{6}$$

By this lemma the electrostatic energy of the configuration considered can be written as a double sum

$$\frac{q^2}{4rn}\sum_{j=1}^{n-1}\sum_{j+1}^{n}\frac{1}{\sqrt{2-2\cos^2\left(\frac{2\,\pi(i-j)}{n}\right)}}.$$

The summation over j can be effectively performed, which yields formula (5).

The fact that this is a non-degenerate minimum follows from the formula for the Hessian matrix of Coulomb energy calculated at regular n-gon [1]. The fact that this is actually global minimum of CH-energy of charged Hookean contour follows by a slight modification of the proof of analogous statement in [2]. As usual non-degeneracy of minimum implies its stability. A rigorous formulation of the final conclusion is given below.

Proposition 3. The ground state configuration of n equal charges on a Hookean contour is non-degenerate and strongly stable, i.e., a small displacement of any single charge increases the CH-energy of system and creates a restoring force at displaced charge.

In conclusion, we add a few words about the inverse problem IEP. The case of two charges is trivial so we proceed by considering the first non-trivial case.

Theorem 4. *Any triangle can serve as a ground state configuration of a certain planar Hookean contour with three charged beads.*

This means that given a triangle we can find parameters q, L, k such that the ground state of the problem N(3, q; L, k) is isometric to the given triangle. This is a direct generalization of analogous result in [4] and the proof can be achieved by the same argument as in [4]. As we already know regular n-gons are possible equilibrium shapes of our model. We can also show that any cyclic quadrilateral is a possible equilibrium shape. In general case problem IEP remains open.

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