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Geophysics

Nonlinear Equations of Surface Gravity Waves

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(Presented by Academy Member Tamaz Chelidze)

ABSTRACT. It is shown that linear and nonlinear theories of surface gravitational waves are incorrect, since they are based on the assumptions of the incompressibility of liquids and the potentiality of their motion, which are not satisfied in the gravitational field of the Earth. Moreover, the incorrectness of these theories is further aggravated by the fact that in order to simplify the solution of the hydrodynamic equations the hydrostatic averaging method is used, with the help of which the phase velocity of both linear and nonlinear surface gravitational wave whose square is equal to the product of the acceleration due to gravity of the reservoir depth is determined. According to new theoretical results, which were published by the author earlier, it is known that the linear theory of surface gravity waves describes only capillary waves, on which, the influence of the Earth's gravitational field is negligible and thus their phase velocity does not depend on the acceleration of gravity. In this paper, nonlinear equations for wind and tsunami waves are obtained, which are devoid of the drawbacks mentioned above. On the basis of logical reasoning, which requires the fulfillment of natural conditions on the water surface, it is shown that the phase velocity of tsunami wave should depend on the acceleration of gravity and depth as well as on the thermodynamic characteristics of air and water. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Keywords: compressibility, incompressibility, potential movement**,** wind wave, tsunami wave

A great number of works of both modern scientists and prominent scientists of the 18-19 centuries were devoted to the problems of surface gravity waves (waves on water or water waves). These waves are generated and propagated on a plane interface between water and air and are described by either linear (when the perturbed values of the thermodynamic parameters of these two media are much smaller, than their equilibrium value) or nonlinear equations of hydrodynamic (when these perturbations are equal to or greater, than their equilibrium value). Accordingly, there are two theories of surface gravity waves: linear and nonlinear. Both these theories are based on two fundamental assumptions: water is an incompressible medium and its motion is potential. We showed the erroneousness of these assumptions back in 2014 in [1] and then in [2] and [3]. They convincingly prove that the linear theory describes only capillary waves, on which gravitational field of the Earth practically does not have effect. Thus the capillary-gravitational waves do not exist in nature. Consequently, there is no condition limiting the length of the capillary wave. As for the nonlinear theory, its erroneousness is further aggravated by the fact that it uses the hydrostatic averaging method, which refers only to the vertical component of the equation of motion, and the change the vertical component of the fluid velocity is neglected [4-8]. In the works mentioned above, we show that application of this method is incorrect, because neglecting the change of the vertical component of velocity leads to an equation for the equilibrium of the liquid, when oscillations are impossible. It is obvious that a theory based on so many incorrect assumptions, cannot give correct results. Readers may object referring to the Korteweg-de Vries equation [5], which despite the presence of all of the listed shortcomings, still gives a completely understandable analytical solution describing solitary waves. In response to this objection, an example of the Kelvin problem of a capillary-gravitational wave [3] may be given, which despite the containing errors in the particular case, when g=0, gives the correct result. The purpose of this paper is to suggest our vision of method for solving a nonlinear problem of surface gravitational waves.

Derivation of the Nonlinear Equation of the Gravitational Wave in Liquids

We write the system of hydrodynamic equations for an ideal fluid in the gravitational field of the Earth in Lagrange coordinates:

$$\begin{cases} \dfrac{d\vec{V}}{dt} = -\dfrac{1}{\circ}\nabla P + \vec{g} \\ \dfrac{d\circ}{dt} = -\circ\nabla\vec{V} - \dfrac{\vec{V}\nabla P}{C_p^2} \end{cases}. \tag{1}$$

We note that the adiabatic equation is already used here, and C_p is the isobaric speed of sound, which according to the new theory [3] can be considered infinitely large for water, and thus the sound velocity in water is adiabatic, i.e. $C = C_s$. The latter means that water is compressible medium and the use of the incompressibility condition $\nabla\vec{V} = 0$ is unacceptable. Since nonlinear waves of wind and tsunami propagate at the interface between water and air, we are dealing with a typical problem of tangential discontinuity and therefore the method of solving it must be similar to the method of solving the problem of linear surface wave. As is known, this method is that the wave equations for each of these two media are solved, and then these solutions are joined on the surface of a tangential discontinuity. Thus, by analogy with the linear theory, we represent all quantities in the form of a sum of their stationary and perturbed values:

$$P = P_0(z) + P'(x,y,z,t), \quad \circ = \circ_0 + \circ'(x,y,z,t), \quad \vec{V} = \vec{V}'(x,y,z,t). \tag{2}$$

i.e.we assume that in a state of equilibrium the liquid is stationary and the perturbed value of pressure satisfies the condition: $P' \geq P_0$. Substituting (2) into (1) and applying the equations of equilibrium and the state of the fluid $\left(\nabla P_0 = \circ_0 g, \circ' = \left(1/C^2\right)P'\right)$, we easily obtain

$$\begin{cases} \dfrac{d\vec{V}}{dt} = \dfrac{P'\vec{g} - C^2\nabla P'}{\circ_0 C^2 + P'} \\ \nabla\vec{V} = -\dfrac{1}{\circ_0 C^2 + P'}\dfrac{dP'}{dt} \end{cases}. \tag{3}$$

We introduce a dimensionless perturbation of the pressure $\overline{P} = P'/\circ_0 C^2$, after which the system (3) takes the form:

$$\begin{cases} \dfrac{d\vec{V}}{dt} = \dfrac{\overline{P}\vec{g} - C^2\nabla\overline{P}}{1+\overline{P}} \\ \nabla\vec{V} = -\dfrac{1}{1+\overline{P}}\dfrac{d\overline{P}}{dt} \end{cases} . \tag{4}$$

Applying the operator ∇ to first equation of system (4) and operator d/dt to the second equation and equating their right-hand sides, we obtain

$$\nabla\overline{P}\left(\vec{g} + C^2\nabla\overline{P}\right) - C^2\Delta\overline{P}\left(1+\overline{P}\right) = \left(\frac{d\overline{P}}{dt}\right)^2 - \left(1+\overline{P}\right)\frac{d^2\overline{P}}{dt^2} . \tag{5}$$

Equation (5) is a nonlinear equation of the gravitational wave in fluid, which, if neglected by nonlinear terms, transforms into the linear equation (62) in [3]. This equation must be supplemented by the first equation of the system (4) and thus we obtain closed system of four equations with respect to four unknowns $V_x, V_y, V_z, \overline{P}$. For water $°{}_0C^2 \cong 2,25\times10^9$ Pa and atmospheric pressure at sea level $P\left(0\right) \cong 10^5$ Pa, and thus it is obvious that the dimensionless pressure perturbation caused by the wind $\overline{P} << 1$, and it should be dropped in comparison with the unit. Tsunami waves are generated as a result of a powerful earthquake, reaching a magnitude of $M \geq 8$ on the Richter scale. Magnitude is associated with the energy released by the earthquake by formula

$$M = \frac{2}{3}\left(\lg E - 4,8\right) \tag{6}$$

and consequently, for the magnitude $M \geq 8$ we get $E \geq 10^{16,2}$ J. If this energy is released in volume $^3 \cong 10^7$ m^3, which is quite real, then a unit should be dropped in comparison with $\overline{P}$. Thus, we have the following two systems of non-linear differential equations for wind and tsunami waves:

$$\text{Wind wave - } \begin{cases} \nabla\overline{P}\left(\vec{g} + C^2\nabla\overline{P}\right) - C^2\Delta\overline{P} = \left(\dfrac{d\overline{P}}{dt}\right)^2 - \dfrac{d^2\overline{P}}{dt^2} \\ \dfrac{d\vec{V}}{dt} = \overline{P}\vec{g} - C^2\nabla\overline{P} \end{cases} \tag{7}$$

$$\text{Tsunami wave - } \begin{cases} \nabla\overline{P}\left(\vec{g} + C^2\nabla\overline{P}\right) - C^2\overline{P}\Delta\overline{P} = \left(\dfrac{d\overline{P}}{dt}\right)^2 - \dfrac{\overline{P}d^2\overline{P}}{dt^2} \\ \dfrac{d\vec{V}}{dt} = \dfrac{\overline{P}\vec{g} - C^2\nabla\overline{P}}{\overline{P}} \end{cases} . \tag{8}$$

Discussion of Methods for Solving the Problem and the Expected Results

Analysis of the literature devoted to the problems of nonlinear gravitational waves on the water surface shows that the results of their analytical and numerical solutions are very approximate and unreliable. This is due to the extreme mathematical complexity of the equations being solved, for the simplification of which unreasonable methods are often used. The situation is further aggravated by the fact that, the entire theory of surface gravitational waves is based on incorrect assumptions. They have been eliminated in the linear theory [1-3], but remain in the nonlinear theory and as long as this is the so, no progress in solving these problems can be expected. Our pessimism is shared by the author of [8, p.2],

In this section, we intend to discuss some of the well-known aspects of the tsunami theory that seem dubious to us and to offer their alternatives. We start with the method of hydrostatic averaging, from which in the linear theory the expression for the phase velocity of the surface gravitational wave $U_p = \sqrt{gh}$ follows, where *h* is the depth of the reservoir. It is noted in [7] that rigorous justification of this approximation was never given, however, it is often used in the nonlinear theory, since by integrating the equation of motion along the vertical coordinate, it allows to express the unknown pressure in the liquid as the sum of terms related to atmospheric pressure and sea level change. It is shown in [1-3] that: firstly, hydrostatic averaging is unacceptable in principle, since it leads to an equilibrium equation, which excludes any vibrational motions in liquid and, secondly, in the linear theory, the gravitational acceleration disappears in connection with the negligent smallness of the gravitational effect at small perturbations and thus, this formula does not follow even from the linear theory. Nevertheless, this expression is widely used in determination of the phase velocity of nonlinear wave, including the tsunami wave. For example, Tables in [7], show tsunami wavelengths, with periods of 5 minutes and 30 minutes for depths from 10000 to 100 meters, calculated by this formula. The article does not say how much these calculations correspond to real measurements (obviously such measurements were not made), but they show a tendency of shortening the wavelength and slowing down as it approaches the shore, i.e. at $h \to 0$, wavelength ª $\to 0$ and phase velocity $U_p \to 0$. This trend is not consistent with reality, because at run-up into the shore, the tsunami wave is accelerated and accumulates huge kinetic energy, which cannot be explained by the destruction of the wave even ten meters or more in height.

It should also be noted that this formula is obtained for water of constant depth and it cannot be applied to the case of variable depth. In the linear theory, the phase velocity of a capillary wave in shallow water is determined by the formula $U_p = k\sqrt{\sim h/{}^{\circ}{}_0}$ [3]. Here, the coefficient of surface tension ~ , depth *h* and density ${}^{\circ}{}_0$ are constants related only to water, i.e. formula does not contain the characteristic values related to air, in spite of the fact that the capillary wave propagates at the interface of these two media. From this formula it is clear that $h \to 0 \Rightarrow U_p \to 0$, which is quite natural, since the perturbations are so small that their appearance on the surface of the Earth does not entail any changes in the air. In the nonlinear theory we are dealing with large perturbations and if we could somehow determine the analytic expression for the phase velocity of a nonlinear wave on the surface of water of constant depth, we assume that it should have the form:

$$U_p = C_1\left[1 - f\left(h, g, {}^{\circ}{}_{01}, {}^{\circ}{}_{02}, C_2\right)\right]. \tag{9}$$

Here, C_1 and ${}^{\circ}{}_{0i}$ $(i = 1;2)$ are the speeds of sound and density in air and water, respectively, and f is certain dimensionless function that varies within $0 \le f < 1$ and satisfies the condition: $f \to 0$ \at $h \to 0$. Such form of phase velocity of a nonlinear surface gravitational wave follows from a completely logical assumption that since the wave propagates at the interface between air and water, its phase velocity must contain the thermodynamic characteristics of these two media and a strong perturbation of the earth's surface $(h = 0)$ should propagate in the air at the speed of sound. It is also important to note that for the origin of tsunami, the determining factor is not the energy but the density of energy released during the earthquake. Perhaps this is the reason that sometimes a sufficiently powerful earthquake does not lead to the emergence of tsunami.

The most relevant for nonlinear waves propagating on the water surface of variable depth is the problem of the evolution of the wave amplitude at run-up into the shore. A brief review of the papers devoted to this problem is given in [9]. It gives the Boussinesq's formula, which predicts that $¦_{\max} \approx 1/h$ and the Green's formula, from which it follows that $¦_{\max} \approx 1/h^{1/4}$, where $¦_{\max}$ and h are the maximum amplitude and variable depth which are normalized to some constant depth d. The author cites the opinion of Miles [10], according to which the Boussineq's formula is valid for a small slope of the coast but takes into account the nonlinearity of the wave, while the Green's formula is valid for linear waves of small amplitude when coasting to shore with a large slope. In our opinion, the validity of these formulas is very doubtful, since they do not take into account the gravitational acceleration, the wavelength and its velocity, i.e. the amplitude of any wave should grow equally.

The physical reason for the increase of the amplitude of the tsunami wave at run-up into the shore is explained as follows [11]: when approaching the shore, the depth of the ocean decreases, and the wave slows down, so that the kinetic energy of the liquid particles distributed vertically concentrates in an ever smaller column of liquid and therefore the wave height increases. We do not exclude that such process really takes place, but in our opinion the main reason is connected with the outflow of water before the tsunami. The explanation of the cause of the outflow also seems unconvincing to us, for it is associated with the failure of the ocean floor during an earthquake. We think that the more convincing explanation of these phenomena is the following: when approaching the shore, the speed of the tsunami wave slows down to a certain depth*, after which it accelerates and thus the second wave lags behind the first. Prior to the approach of the second wave, the first wave has time to move away from the shore (outflow) and strike the second wave under arm, reducing its acceleration, in consequence of which its amplitude rises sharply.

Conclusion

This article offers a qualitatively new approach to the problems of nonlinear surface gravity waves in general and to tsunami waves in particular.We understand that experts using traditional methods may reject it, so as it does not simplify, but rather complicates the solution of the problem. The complication lies not only in the fact that we do not use the conditions of incompressibility of water and the potentiality of its flow, as well as the method of static averaging, but in a greater degree by the fact that we are proposing to solve a nonlinear differential equation with respect to the pressure for two media and to join these solutions on their boundary.

გეოფიზიკა

ზედაპირული გრავიტაციული ტალღების არაწრფივი განტოლებები

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(წარმოდგენილია აკადემიის წევრის თ. ჭელიძის მიერ)

ნაჩვენებია, რომ ზედაპირული გრავიტაციული ტალღების წრფივი და არაწრფივი თეორიები არაკორექტულია, ვინაიდან ისინი ემყარებიან მოსაზრებებს სითხეთა უკუმშვადობისა და მათი მოძრაობის პოტენციურობის შესახებ, რომლებიც არ სრულდება დედამიწის გრავიტაციულ ველში. გარდა ამისა, მათ არაკორექტულობას აღრმავებს ისიც, რომ ჰიდროდინამიკურ განტოლებათა სისტემის ამოხსნის გამარტივების მიზნით, მათში გამოიყენება ჰიდროსტატიკური გასაშუალოების მეთოდი, რომლის საშუალებითაც განისაზღვრება წრფივი და არაწრფივი ზედაპირული გრავიტაციული ტალღის ფაზური სიჩქარე, რომლის კვადრატი ტოლია სიმძიმის ძალის აჩქარების და წყლის სიღრმის ნამრავლისა. ახალი თეორიული შედეგების თანახმად ცნობილია, რომ წრფივი თეორია აღწერს მხოლოდ კაპილარულ ტალღებს, რომლებზეც დედამიწის გრავიტაციული ველის გავლენა უმნიშვნელოა და ამდენად, მათი ფაზური სიჩქარე არ შეიცავს სიმძიმის ძალის აჩქარებას. წარმოდგენილ ნაშრომში მიღებულია ქარისა და ცუნამის ტალღების წრფივი და არაწრფივი განტოლებები, რომლებიც არ ემყარებიან აღნიშნულ წინააღმდეგობრივ მოსაზრებებს. ლოგიკური მსჯელობის საფუძველზე, რომელიც ითხოვს წყლის ზედაპირზე ბუნებრივი პირობების დაკმაყოფილებას, ნაჩვენებია, რომ ცუნამის ტალღის ფაზური სიჩქარე დამოკიდებული უნდა იყოს სიმძიმის ძალის აჩქარებასა და ოკეანის სიღრმეზე და ასევე წყლისა და ჰაერის თერმოდინამიკურ მახასიათებლებზე.

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