Mathematics

The Riemann-Hilbert Boundary Value Problem for Carleman-Vekua Equation with Polar Singularities

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ABSTRACT. The paper deals with the Riemann-Hilbert boundary value problem for the Carleman-Vekua equation with polar singularities. Index formula and criteria of solvability are obtained. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *Carleman-Vekua equation, polar singularity,boundary value problem.*

In the present work for the Carleman-Vekua equation

$$\frac{\partial w}{\partial \bar{z}} + A(z)w + B(z)\bar{w} = 0 \tag{C-V}$$

with the polar singularities the Riemann-Hilbert boundary value problem

$$Re\left\{\overline{\lambda(t)}w(t)\right\} = \gamma(t), t \in \Gamma, \tag{R-H}$$

in the domain G is investigated. G is a finite $m+1$-connected domain of the complex plane $z = x + iy$ with sufficiently smooth boundary provided that the given functions $\lambda(t)$ and $\gamma(t)$ are the Holder continuous functions.

It is well-known that the equation (C-V) in case of regular coefficients (i.e. $A(z), B(z) \in L_p(G)$ for some $p > 2$) the condition $\lambda(t) \neq 0, t \in \Gamma$ provides the Noetherity of the problem (R-H) in the class of continuous in $\bar{G} \setminus \{z_0\}$ functions satisfying the equation (C-V) in $G \setminus \{z_0\}$ and the asymptotic conditions $O\left(|z - z_0|^{\sigma}\right), z \to z_0$. Here z_0 is some point in the domain G and σ is some real number.

For the Carleman-Vekua equation with the polar singularities the situation is essentially different. It is known that there exists sufficiently wide class of equations permitting only trivial solutions in the domain $G \setminus \{z_0\}$ and satisfying the asymptotic conditions $O\left(|z - z_0|^{\sigma}\right), z \to z_0$, where z_0 is the point of polar singularity of the equation (C-V), σ is a real number. Therefore it makes no sense to consider the boundary value problems in this class. On the other hand if there are no restrictions on the solutions in the neighborhood of the singular point z_0 then it may occur that the homogeneous boundary problem has infinite number of linearly independent solutions.

In this work for the Riemann-Hilbert problem the Noetherity conditions in particular like asymptotic conditions $O\left(exp\left\{\delta_0\left|z-z_0\right|^{-\sigma_0}\right\}\right), z\to z_o$ for sufficiently wide class of the Carleman-Vekua equations are obtained. Here the constant parameters δ_0, σ_0 are uniquely defined by means of the coefficients of the equation, are independent from the given boundary functions and characterize the polar singularities of the coefficients. These asymptotic conditions are in some sense exact since if we seek the solution of the Riemann-Hilbert problem in the class satisfying the asymptotic condition $O\left(exp\left\{\delta\left|z-z_0\right|^{-\sigma}\right\}\right), z\to z_{0,}$ and if at least one from the equalities $\delta=\delta_0, \sigma=\sigma_0$ is not fulfilled then either the homogeneous problem has infinite number of linearly independent solutions or the non-homogeneous problem is not solvable for any right-hand side.

In section 1 the abovementioned asymptotic conditions are obtained; the general representation of the solutions of the Carleman-Vekua equations with the polar singularities satisfying these conditions are constructed. By means of these results the Riemann-Hilbert problem is correctly posed and is completely investigated in section 2.

1. The Carlemann-Vekua equations with the polar singularities.

1^0. Let G be a bounded complex domain with the boundary Γ consisting from closed non-intersecting Liapunov smooth $\Gamma_0,\Gamma_1,\ldots,\Gamma_m$ contours and Γ_0 covers all the rest.Let G^* be some finite subset of the set G;

$$G^*=\left\{z_1,z_2,\ldots,z_N\right\}, N\geq 1.$$

Consider the Carlemann-Vekua equation

$$\frac{\partial w}{\partial\bar{z}}+A(z)w+B(z)\overline{w}=0, \tag{1.1}$$

in the domain G, provided that the coefficient $B(z)\in L_p(G), p>2$ and the coefficient $A(z)$ admits the following representation

$$A(z)=\overline{g(z)}+\sum_{k=1}^{N}\frac{A_k(z)}{\left|z-z_k\right|^{\nu_k}}, \tag{1.2}$$

where the function $g(z)$ is holomorphic in $G\setminus G^*$ and has continuous boundary value on Γ; the function $A_k(z)$ admits the following representation

$$A_k(z)=a_k(z)exp\left\{in_k arg\left(z-z_k\right)\right\}, \tag{1.3}$$

where

$$\frac{a_k(z)-\lambda_k}{\left|z-z_k\right|^{\nu_k}}\in L_p(G), p>2;$$

the constants λ_k,ν_k,n_k are correspondingly complex, positive and entire numbers for every $k=1,2,..,N\left(cp.[1],[2]\right)$.

Under the solution of the equation (1.1) is understood the continuous generalized solution in $G\setminus G^*$; denote by $\mathfrak{R}\left(A,B,G\setminus G^*\right)$ the set of all possible such solutions (cp. [5]).

2^0. Everywhere below the fulfillment of the following condition

$$\lambda_k\neq 0;\left|n_k-1\right|>2\left(\nu_k-1\right)>0, k=1,2,\ldots,N \tag{1.4}$$

is assumed.

As seen from (1.2) the coefficient $A(z)$ has the polar singularities of the form $\left|z-z_k\right|^{-\nu_k}$ and the singularities of the function $\overline{g(z)}$ in the points z_k. Below it will be established that the structure of the solutions of the Carleman-Vekua equation depends on the relationship between the parameters λ_k, ν_k, n_k and if the required conditions between them are violated then, generally speaking, the assumptions proved below are not valid.

3^0. The following notations we need below

$$q_k = Res_{z=z_k}^{g(z)}, k = 1,2,\ldots,N; Q_k = \frac{1}{2\pi i}\int_{\Gamma_k} g(t)\,dt, k = 1,2,\ldots,m.$$

Introduce an auxiliary function given by the formula

$$f(z) = \int_{\Gamma_{\varsigma_0,z}} \overline{g(t)dt} - \sum_{k=1}^{m} \overline{Q_k}\log\left(\overline{z} - \overline{\tau_k}\right) - \sum_{k=1}^{N}\overline{q_k}\log\left(\overline{z} - \overline{z_k}\right),$$

in the domain $G \setminus G^*$, where ς_0 is some fixed point in $G \setminus G^*; \Gamma_{\varsigma_0,z}$ is a smooth contour connecting the points ς_0, z and lying in $G \setminus G^*; \tau_k$ is an arbitrary fixed point inside the contour $\Gamma_k, k = 1,2,\ldots,m.$ Consider also the function

$$F(z) = \Lambda(z) exp\left\{2iImf(z)\right\}\chi(z), z \in G \setminus G^*, \tag{1.5}$$

where

$$\Lambda(z) = \prod_{k=1}^{N}(z - z_k)^{-[2Req_k]}, \tag{1.6}$$

$$\chi(z) = exp\left\{2\sum_{k=1}^{m} Q_k \log|z - \tau_k| + 2\sum_{k=1}^{N} q_k \log|z - z_k|\right\}. \tag{1.7}$$

It follows from the conditions (1.4) that $2 - \nu_k - n_k \neq 0, k = 1,2,\ldots,N$ and therefore by the formulas

$$\delta_k^* = \frac{2\lambda_k}{2 - \nu_k - n_k}, k = 1,2,\ldots,N \tag{1.8}$$

the definite non-zero numbers are given. Assume

$$R(z) \equiv \sum_{k=1}^{N}\frac{\delta_k^*}{|z - z_k|^{\nu_{k-1}}} \bullet exp\left\{i(n_k - 1)\bullet arg(z - z_k)\right\}, \tag{1.9}$$

$$\Psi(z) \equiv F(z) exp\left\{R(z)\right\}. \tag{1.10}$$

Consider the following Carlenam-Vekua equation

$$\frac{\partial w_*}{\partial \overline{z}} + A_*(z) w_* + B_*(z)\overline{w_*} = 0, \tag{1.11}$$

where

$$A_*(z) = \sum_{k=1}^{N}\frac{a_k(z) - \lambda_k}{|z - z_k|^{\nu_k}} e^{in_k arg(z - z_k)},\ B_*(z) = \frac{B(z)\Psi(z)}{\overline{\Psi(z)}}.$$

It is easy to see, that $A_*(z), B_*(z) \in L_p(G), p > 2$ and hence (1.11) is the regular Carleman-Vekua equation.

The following theorem takes place.

Theorem 1.1. *By the following relation*

$$w_*(z)=\Psi(z)w(z), z\in G\setminus G^* \tag{1.12}$$

$$\left(w_*\in\Re\left(A_*,B_*,\ G\setminus G^*\right), w\in\Re\left(A,B,G\setminus G^*\right)\right),$$

the bijective correspondence between the classes $\Re\left(A,B,G\setminus G^*\right)$ and $\Re\left(A_*,B_*,\ G\setminus G^*\right)$ is established.

Proof. One can check directly the following inequalities

$$\frac{\partial F(z)}{\partial\bar{z}}=F(z)\bullet\overline{g(z)},\ \frac{\partial exp\{R(z)\}}{\partial\bar{z}}=exp\{R(z)\}\sum_{k=1}^{N}\frac{\lambda_k}{|z-z_k|^{\nu_k}}\bullet exp\{in_k\bullet arg(z-z_k)\},$$

$$\frac{\partial\Psi(z)}{\partial\bar{z}}=\Psi(z)\left[\sum_{k=1}^{N}\frac{\lambda_k}{|z-z_k|^{\nu_k}}\bullet exp\{in_k\bullet arg(z-z_k)\}+\overline{g(z)}\right].$$

It is clear that by means of the relation (1.12) the bijective correspondence is also established between the classes

$$\Re\left(A_*,B_*,G\setminus G^*\right)\bigcap C\left(\bar{G}\setminus G^*\right),\quad \Re\left(A,B,G\setminus G^*\right)\bigcap C\left(\bar{G}\setminus G^*\right).$$

4^0. Let $\delta=(\delta_1,\delta_2,\ldots,\delta_N)$ and $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_N)$ are given N -dimensional vectors with the non-nega-tive components. Denote by $\Omega_0[\delta,\sigma]$ the class of all possible functions from the set $\Re\left(A,B,G\setminus G^*\right)$ satisfying the condition

$$w(z)=O\left(exp\left\{\delta_k|z-z_k|^{-\sigma_k}\right\}\right), z\to z_k, k=1,2,\ldots,N. \tag{1.13}$$

Denote by $\overline{\Omega_0}[\delta,\sigma]$ the class of all possible functions from the set $\Omega_0[\delta,\sigma]$ admitting the continuous extension in $\left(\bar{G}\setminus G^*\right)$. By δ^* and ν^* the following vectors are denoted

$$\delta^*\equiv\left(\left|\delta_1^*\right|,\left|\delta_2^*\right|,\ldots,\left|\delta_N^*\right|\right), \nu^*\equiv(\nu_1-1,\nu_2-1,\ldots,\nu_N-1).$$

The class of the solutions $\Omega_0\left[\delta^*,\nu^*\right]$ is very important class in what follows.

The following theorem occurs.

Theorem 1.2. *If for some value k of the index the inequality $\delta_k<\left|\delta_k^*\right|$ is fulfilled then the class $\Omega_0\left[\delta,\nu^*\right]$ is a trivial class (i.e. it contains only zero functions).*

The proof of the theorem 1.2 follows easily from the works [3], [4].

Straight from the theorem 1.2 it follows that if for some value k of the index the inequality $\sigma_k<\nu_k-1$ is fulfilled then for every vector δ the class $\Omega_0[\delta,\sigma]$ is a trivial class.

Therefore, if $\sigma_k=\nu_k-1, k=1,2,\ldots,N$ and for some k_0 the inequality $\delta_{k_0}<\left|\delta_{k_0}^*\right|$ is fulfilled or if for some k_0 the inequality $\sigma_{k_0}<\nu_{k_0}-1$ is fulfilled then the class $\Omega_0[\delta,\sigma]$ is a trivial class.

It is natural to investigate the class $\Omega_0[\delta,\sigma]$. The following theorem gives us the representation of the solution of this class.

Theorem 1.3. *By means of the relation (1.3) the bijective correspondence between the clas-ses*

$$\Omega_0\left[\delta^*,\nu^*\right]\left(\overline{\Omega_0}\left[\delta^*,\nu^*\right]\right),\ \Re(A_*,B_*,G)\left(\Re(A_*,B_*,G)\bigcap C(\bar{G})\right)$$

is established.

2. Investigation of the Riemann-Hilbert boundary value problem.

In order to pose correctly the Riemann-Hilbert boundary value problem it is clear from the abovemen-tioned results that it is sufficient to require from the solution of the equation (1.1) the fulfillment of the asym-ptotic condition of the form

$$w(z)=O\left(exp\left\{\left|\delta_k^*\right|\bullet\left|z-z_k\right|^{-\nu_k+1}\right\}\right), z\to z_k, k=1,2\ldots,N. \tag{*}$$

In the present section the proof of sufficiency of asymptotic condition (*) is given and the boundary value problems are investigated.

Consider the following boundary value problem: on the boundary Γ the Holder continuous functions $\lambda(t)$ and $\gamma(t)$ are given, $\gamma(t)$ is a real function and $|\lambda(t)|=1$; find the function $w(z)\in\overline{\Omega_0}[\delta,\sigma]$ satisfying the boundary equation

$$Re\left\{\overline{\lambda(t)}w(t)\right\}=\gamma(t),\quad t\in\Gamma. \tag{2.1}$$

From the theorem 1.2 it follows that in case $\gamma(t)\neq 0$ the problem (2.1) is not solvable in the class $\overline{\Omega_0}[\delta,\sigma]$ if $\sigma_k=\nu_k-1$ for some k or if $\sigma_k=\nu_k-1, k=1,2,\ldots,N,$ but $\delta_k<\left|\delta_k^*\right|$ for some k.
Let $\delta_k\neq 0, \sigma_k\geq\nu_k-1, k=1,2,\ldots,N$. Denote by $\mathcal{H}$ the set of all possible values k of the index for which $\delta_k<\left|\delta_k^*\right|$.

The following theorem takes place.

Theorem 2.1. *The homogeneous boundary value problem (2.1)* $(\gamma(t)=0)$ *in the class* $\overline{\Omega_0}[\delta,\sigma]$ *has infinite number of linearly independent solutions if and only if when one from the following conditions is fulfilled:*

1) $\mathcal{H}=\varnothing; \sum_{k=1}^{N}(\delta_k+\sigma_k)>\sum_{k=1}^{N}\left(\left|\delta_k^*\right|+\nu_k-1\right);$

2) $\mathcal{H}\neq\varnothing; \sigma_k>\nu_k-1, k\in\mathcal{H}.$

Proof. Let the first condition 1) be fulfilled. Then for all $k=1,2,\ldots,N$ the following inequalities hold $\delta_k\geq\left|\delta_k^*\right|+\nu_k-1,$ and there exists at least one $k=k_0$ for which the strict inequality is fulfilled

$$\delta_{k_0}+\sigma_{k_0}>\left|\delta_{k_0}^*\right|+\nu_{k_0}-1, \tag{2.2}$$

From this last inequality follows that one from the inequalities $\delta_{k_0}\geq\left|\delta_{k_0}^*\right|, \sigma_{k_0}\geq\nu_{k_0}-1$ is also strict. Let $\delta_{k_0}>\left|\delta_{k_0}^*\right|$ and let us fix an arbitrary number $S\geq 0$. Consider the solutions $w_*(z)$ from the class $\mathfrak{R}\left(A_*,B_*,G\setminus G^*\right)$ which are representable in the form

$$w_*(z)=\frac{\Phi(z)}{\left(z-z_{k_0}\right)^s}exp(\omega(z)), \tag{2.3}$$

where $\Phi(z)$ is holomorphic in G and continuous in $\bar{G}$ function. It is easy to see that every function of the form (2.3) defines the solution $w(z)$ of the equation (1.1) of the class $\overline{\Omega_0}[\delta,\sigma]$ by means of the relation (1.12). Further on, we can see that by the relation

$$w_*(z)=\frac{\omega_0(z)}{\left(z-z_{k_0}\right)^s} \tag{2.4}$$

the bijective correspondence between the class of all functions of the form (2.3) and the class

$$\mathfrak{R}\left(A_*,B_*,\left(\frac{z-z_{k_0}}{\overline{\overline{z}-z_{k_0}}}\right)^s,G\right)\bigcap C\left(\bar{G}\right)$$

of the solutions of the equation

$$\frac{\partial\omega_0}{\partial\overline{z}} + A_*(z)\omega_0 + B_*\left(\frac{z - z_{k_0}}{\overline{z} - \overline{z_{k_0}}}\right)^s \overline{\omega_0} = 0. \tag{2.5}$$

is established.

Together with the problem (2.1) consider the following boundary value problem: find the generalized solution of the problem (2.5) continuous in G and satisfying the boundary condition

$$Re\left\{\frac{\overline{\lambda(t)}}{\left(t - z_{k_0}\right)^s \Psi(t)}\omega_0(t)\right\} = 0, \quad t \in \Gamma. \tag{2.6}$$

It is clear that by the formulas (2.4), (1.12) every system of linearly independent solutions of the problem (2.6) defines the system of linearly independent solutions of homogeneous problem (2.1). On the other hand the number of linearly independent solutions of the problem (2.6) l satisfies the inequality

$$l \geq 2ind\left(\frac{\lambda(t)}{\left(\overline{t} - \overline{z_{k_0}}\right)^s \overline{\Psi(t)}}\right) - m + 1 \tag{2.7}$$

by virtue of which we get

$$l \geq 2\left(ind\lambda(t) + S + \sum_{k=1}^{N}\left[2Req_k\right]\right) - m + 1.$$

From here it follows that if an appropriate choice of S the number l will be arbitrarily big and therefore the homogeneous problem (2.1) has infinite number of linearly independent solutions. One can prove similarly that the set of linearly independent solutions of the homogeneous problem (2.1) is infinite in case $\sigma_{k_0} > \nu_{k_0} - 1, \delta_{k_0} = \left|\delta^*_{k_0}\right|$. Hence we obtain that if the condition 1) is fulfilled then the homogene-ous problem (2.1) has the infinite number of linearly independent solutions.

Let now the condition 2) be fulfilled. Then on the basis of the relation

$$\overline{\Omega_0}\left[\delta^{(1)}, \sigma^{(1)}\right] \subset \overline{\Omega_0}\left[\delta^{(2)}, \sigma^{(2)}\right],$$

which follows directly from the conditions

$$\delta_k^{(2)} \neq 0, k = 1, 2, \ldots, N; \sigma_k^{(2)} > \sigma_k^{(1)}, k \in \mathcal{H}.$$

we get that the homogeneous problem (2.1) has infinite number of linearly independent solutions. The sufficiency of one of the conditions 1), 2) is proved. Let us prove the necessity.

Let the homogeneous problem (2.1) have infinite number of linearly independent solutions and the set $\mathcal{H} \neq \varnothing$, then for every $k \in \mathcal{H}$ we have $\sigma_k > \nu_k - 1$. Indeed, if for at least one $k_0 \in \mathcal{H}$ we have $\sigma_{k_0} > \nu_{k_0} - 1$, then on the basis of the theorem 1.2 the class $\overline{\Omega_0}\left[\delta, \sigma\right]$ would consist from only zero elements; we get a contradiction and so, when $\mathcal{H} \neq \varnothing$ the condition 2) is fulfilled.

Let now $\mathcal{H} = \varnothing$, then prove that

$$\sum_{k=1}^{N}(\delta_k - \sigma_k) > \sum_{k=1}^{N}\left(\left|\delta_k^*\right| + \nu_k - 1\right) \tag{2.8}$$

Indeed, otherwise it would be

$$\sum_{k=1}^{N}\left(\delta_k+\sigma_k\right)=\sum_{k=1}^{N}\left(\left|\delta_k^*\right|+\nu_k-1\right) \tag{2.9}$$

and therefore $\delta_k=\left|\delta_k^*\right|, \sigma_k=\nu_k-1, k=1,2,\ldots,N.$. Hence we would obtain that the homogeneous problem (2.1) has infinite number of linearly independent solutions in the class $\overline{\Omega_0}\left[\delta^*,\nu^*\right]$, but this problem has finite number of linearly independent solutions. Really, by virtue of the theorem 1.3 using the relation (1.12) the bijective correspondencies established between the solutions of the problem (2.1) and the following boundary value problem: find the generalized solution of the equation

$$\frac{\partial w_*}{\partial \overline{z}}+A_*\left(z\right)w_*\left(t\right)+B_*\overline{w_*}=0, \tag{2.10}$$

continuous in the domain $\overline{G}$ satisfying the boundary condition

$$Re\left\{\frac{\overline{\lambda\left(t\right)}}{\Psi\left(t\right)}w_*\left(t\right)\right\}=\gamma\left(t\right),\ t\in\Gamma. \tag{2.11}$$

The homogeneous problem (2.10)-(2.11) $\left(\gamma\left(t\right)=0\right)$ on the basis of [5] has finite number of linearly independent solutions. Hence, the homogeneous problem (2.1) has finite number of linearly independent solutions in the class $\overline{\Omega_0}\left[\delta^*,\nu^*\right]$. Therefore, the condition (2.9) is not fulfilled and thus (2.8) is fulfilled. The theorem 2.1 is completely proved.

Consider the boundary value problem: find the generalized solution continuous in the class $\overline{G}$ of the equation

$$\frac{\partial w_*^{'}}{\partial \overline{z}}-A_*\left(z\right)w_*^{'}\left(t\right)-\overline{B_*\left(z\right)}\overline{w_*^{'}}=0, \tag{2.12}$$

satisfying the boundary condition

$$Re\left\{\frac{\lambda\left(t\right)}{\overline{\Psi\left(t\right)}}t^{'}\left(s\right)w_*^{'}\left(t\right)\right\}=0, t\in\Gamma. \tag{2.13}$$

It is easy to see that the number of linearly independent solutions of the problem (2.13) l_* is finite and it is clear that for the problem (2.10) to be solvable it is necessary and sufficient the fulfillment of equality

$$\int_{\Gamma}\frac{\lambda\left(t\right)}{\overline{\Psi\left(t\right)}}\gamma\left(t\right)w_*^{'}\left(t\right)dt=0 \tag{2.14}$$

for every solution of the problem (2.13).

On the basis of above obtained results the following theorem becomes evident.

Theorem 2.2. *The homogeneous problem (2.1) in the class* $\overline{\Omega_0}\left[\delta^*,\nu^*\right]$ *has finite number of linearly independent solutions and the non-homogeneous problem is solvable if and only if the condition (2.14) is fulfilled.*

Let l be a number of linearly independent solutions of the homogeneous problem (2.1). By means of following evident equality

$$ind\left(\frac{1}{\overline{\Psi\left(t\right)}}\right)=\sum_{k=1}^{N}\left[2Req_k\right]$$

it follows the validity of the next problem.

Theorem 2.3. *The following equality*

$$l - l_* = 2n + 2\sum_{k=1}^{N}\left[2Re q_k\right] - m + 1$$

takes place, where q_k is a residue of the function $q(z)$ in the point z_k.

From the results obtained above in particular we get the theorem.

Theorem 2.4. *For the problem (2.1) to be Noetherian in the class* $\overline{\Omega_0}\left[\delta, \sigma\right]$ *it is necessary and sufficient to fulfill the condition*

$$\delta = \delta^*, \sigma = \nu^*.$$

მათემატიკა

რიმან-ჰილბერტის სასაზღვრო ამოცანა კარლემან-ვეკუას განტოლებისათვის პოლარული განსაკუთრებულობებით

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** საქართველოს საპატრიარქოს წმიდა ანდრია პირველწოდებულის სახელობის ქართული უნივერსიტეტი, თბილისი*

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ნაშრომში განხილულია რიმან-ჰილბერტის სასაზღვრო ამოცანა კარლემან-ვეკუას განტოლებისათვის პოლარული განსაკუთრებულობებით.

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