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Theory of Elasticity

Approximate Solution of Anti-Plane Problem of Elasticity Theory for Composite Bodies Weakened by Cracks by Integral Equation Method

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ABSTRACT. In the present article an anti-plane problem of the elasticity theory for a composite (piece-wise homogeneous) orthotropic body weakened by cracks intersecting the interface or reaching it in a right angle is studied. The studied problem is reduced to the singular integral equation (when crack reaches the interface) and system (pair) of singular integral equations (when crack intersects the interface) containing an immovable singularity with respect to the unknown characteristic function of the crack disclosure. Behavior of solutions in the neighborhood of the crack endpoints is studied by the method of discrete singularity with uniform division of an interval by knots.In both cases (crack intersects or reaches the interface) the question of behavior of approximate solutions are investigated.The corresponding algorithms are composed and realized. The results of numerical investigations are presented. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *singular integral equations, crack, anti-plane problem, method integral equation, stress intensity factor, collocation method, method of discrete singularity.*

Two dimensional problems of cracks of elasticity theory have thoroughly studied in the monographs [1-4]. Study of boundary value problems for the composite bodies weakened by cracks has a great practical significance [5-7]. Approximate solution of the problems of cracks theory was widely studied by collocation and spectral methods [8-10]. In the present article the problem of composite (piece-wise homogeneous) bodies with the cracks intersecting an interface or penetrating it in a right angle is studied. The anti-plane problem of the elasticity theory for piece-wise homogenous orthotropic plane is reduced to the system pair of singular integral equations containing an immovable singularity with respect to the tangent stress jumps. The behavior of solutions in the neighborhood of the crack end points is studied. A general scheme of approximate solutions by the collocation method is composed and realized.

Statement of the Problem

Let elastic Ω body occupy complex variable plane $z = x + i\,y$, which is cut on the line L = [-1;+1]. The plane consists of two orthotropic homogeneous semiplanes $\Omega_1 = \{z : Rez \geq 0, x \notin L_1 = [0;1]\}$ and $\Omega_2 = \{z : Rez \leq 0, x \notin L_2 = [-1;0]\}$,which are welded on the axis y. Define the values and functions connected with Ω_k by the index k, k = 1, 2. The problem is to find the function $w_k(x,y)$, which satisfies the following differential equation and boundary conditions:

$$\frac{\partial^2 w_k(x,y)}{\partial x^2} + \lambda_k^2 \frac{\partial^2 w_k(x,y)}{\partial y^2} = 0,\ (x,y)\epsilon\Omega_k, \tag{1}$$

a) on the boundary of the crack the tangent stresses are given:

$$b_{44}^{(k)} \frac{\partial w_k(x,\pm 0)}{\partial y} = q_k^{(\pm)}(x),\ x\epsilon \mathrm{L}_k \tag{2}$$

b) on the axis y the condition of continuity is fulfilled:

$$w_1(0;y) = w_2(0;y),\ y \in (-\infty;\infty), y \neq 0 \tag{3}$$

$$b_{55}^{(1)} \frac{\partial w_1(0;y)}{\partial x} = b_{55}^{(2)} \frac{\partial w_2(0;y)}{\partial x} \tag{4}$$

where $\lambda_k^2 = \dfrac{b_{44}^{(k)}}{b_{55}^{(k)}}$, $b_{44}^{(k)}, b_{55}^{(k)}$ are elastic constants, $q_k(x)$ is a function of Holder's class, $k = 1,2,$ in particular, if we have isotropic case $b_{44}^{(k)} = b_{55}^{(k)} = \mu_k$, $\lambda_k = 1,$ where μ_k is the module of displacement, $k = 1,2$. We consider a case of symmetric load $q_k^{(+)}(x) = q_k^{(-)}(x) = q_k(x)$, then we have an avoided singularity and we can use the method of integral equations ([1]).Using the theory of analytical functions from boundary conditions (2)-(4) the system of singular integral equations with respect to leaps $\rho_k(x)$ (see [11])

$$\begin{gathered}
\int_0^1 \left(\frac{1}{t-x} - \frac{a_1}{t+x}\right)\rho_1(t)\,dt + b_1 \int_{-1}^0 \frac{\rho_2(t)\,dt}{t-x} = 2\pi f_1(x),\ x\epsilon(0;1), \\
b_2 \int_0^1 \frac{\rho_1(t)\,dt}{t-x} + \int_{-1}^0 \left(\frac{1}{t-x} - \frac{a_2}{t+x}\right)\rho_2(t)\,dt = 2\pi f_2(x),\ x\epsilon(-1;0),
\end{gathered} \tag{5}$$

where $\rho_k(x)$, $f_k(x)$ are unknown and given real functions, respectively; a_k, b_k are constants:

$$a_k = \frac{1-\gamma_k}{1+\gamma_k},\ b_k = \frac{2}{1+\gamma_k},\ \gamma_1 = 1/\gamma_2,\ \gamma_2 = \frac{b_{55}^{(2)}}{b_{55}^{(1)}},\ f_k(x) = \frac{\lambda_k}{b_{44}^{(k)}} q_k(x),$$

$$f_k(x)\epsilon H, \rho_k(x)\epsilon H^*,\ k = 1,2.$$

Explanation of the behavior of solutions near the ends of the boundary is of a special interest. The solutions of the system (5) of the integral equations can be given by:

$$\rho_1(t) = \frac{\chi_1(t)}{t^{\alpha_1}(1-t)^{\beta_1}},\ \rho_2(t) = \frac{\chi_2(t)}{t^{\alpha_2}(1+t)^{\beta_2}} \tag{6}$$

where α_k, β_k are unknown constants $0 < \alpha_k, \beta_k < 1$, and $\chi_k(t)$ are functions, which belong to Holder's class, $k = 1,2$. Correspondingly, in the point $t = \pm 1$ we obtain $\beta_1 = \beta_2 = \dfrac{1}{2}$. In the considered case there is no peculiarity in the point $t = 0$ (see [11]). In a partial case when one semi-plane has a rectilinear cut of finite length, which is perpendicular to the boundary, and one end of which is located on the boundary, we have one singular integral equation containing an immovable singularity. In a partial case, when a crack reaches the boundary of separation, we get that an order of peculiarity in the point $t = 0$ depends on elastic constants of material and belongs to $0 < \alpha < 1$. If $\rho_2(x) \equiv 0,\ \rho_1(x) \neq 0$, then integral of the system (5) gives one integral equation:

$$\int_0^1 \left(\frac{1}{t-x} - \frac{a_1}{t+x} \right) \rho_1(t)\,dt = 2\pi f_1(x), \qquad x\epsilon[0;1] \tag{7}$$

We get that an order of peculiarity in the point $t = 0$ depends on elastic constants of material and belongs to interval (0;1). $\alpha = 1 - \frac{1}{\pi} arccos\left(\frac{b_{55}^{(1)} - b_{55}^{(2)}}{b_{55}^{(1)} + b_{55}^{(2)}} \right) \epsilon(0;1)$, $\beta = \frac{1}{2}$. If $b_{55}^{(1)} = b_{55}^{(2)}$, then $\alpha = \frac{1}{2}$. (see [11]).

On Approximate Solution of Singular Integral Equation by Collocation Method

Anti-plane problems of elasticity theory, composed for orthotropic plane weakened with cracks, are reduced to the following integral equation containing an immovable singularity:

$$\int_0^1 \left(\frac{1}{t-x} - \frac{a}{t+x} \right) \rho(t)\,dt = 2\pi f(x), \quad x\epsilon[0;1] \tag{8}$$

where unknown function $\rho(t) \in H^*([0,1])$, parameter $a\epsilon[-1;1]$ depends on the material's elasticity constants, right side $f(x) \in H_\mu[0,1],\ 0 < \mu \le 1$.

As at the both sides of the body we have infinite solutions, for existence of the uniqueness solution of the decision (8) the following additional condition is necessary [1].

$$\int_0^1 \rho(t)\,dt = C, \tag{9}$$

where C=const is to be chosen.

The analysis of the above-mentioned integral equation and study of their exact and approximate solvution methods are accompanied with some specific complexities due to the fact that the solution has a composite asymptotics, which can be considered only in certain cases introducing weight functions. If we have square root type singularity on both ends of the integral, then Chebishev orthogonal polynoms can be used [6, 8].The integral is replaced with the quadrature formulas of open type [10, 12]. In the work equation system (8)-(9) can be solved by collocation method [9]. Let us enter such distribution of knots for variables of integration and account points accordingly: $t_i = 0 + ih,\quad i = 1,2,\ldots,n;\quad x_j = t_j + \frac{h}{2},\quad , j = 1,2,\ldots,n-1;$ $h = \frac{1}{n+1}$. The equations (8) can be presented as follows with the help quadrature formulas [9],

$$\begin{gathered} \sum_{i=1}^{n} \left(\frac{h}{t_i - x_j} - \frac{a_1 h}{t_i + x_j} \right) \rho\ (t_i) = 2\pi f(x_j), j = 1,2,\ldots,n-1; \\ \sum_{i=1}^{n} h\,\rho\ (t_i) = C, \end{gathered} \tag{10}$$

As well as in the previous case we receive a system of the linear algebraic equations of an order *n* in knots of a grid for definition of the values of an unknown function.

An Approximate Solution of the System of Singular Integral Equations

Now let's consider the system of the singular integral equations (5) containing an immovable singularity [13]. The system (5) of singular integral equations is solved by a collocation method, in particular, by the method of discrete singular (see [9]). First of all, let us consider an algorithm of uniform division. Let us enter such distribution of knots for variables of integration and account points accordingly: $t1_i = 0 + ih,\ t2_i = -1 + ih,\ i = 1,2,\ldots,n; x1_j = t1_j - h/2,\ \ x2_j = t2_j + h/2,\ j = 1,2,\ldots,n; h = \frac{1}{n+1}$. With the help of quadrature formulas [9] the pair of the equations can be given by:

$$\sum_{i=1}^{n}\left(\frac{h}{t1_i - x1_j} - \frac{a_1 h}{t1_i + x1_j}\right)\rho_1(t1_i) + b_1\sum_{i=1}^{n}\left(\frac{h}{t2_i - x1_j}\right)\rho_2(t2_i) = 2\pi f_1(x1_j),$$

$$j = 1,2,\ldots,n;$$

$$b_2\sum_{i=1}^{n}\left(\frac{h}{t1_i - x2_j}\right)\rho_1(t1_i) + \sum_{i=1}^{n}\left(\frac{h}{t2_i - x2_j} - \frac{a_2 h}{t2_i + x2_j}\right)\rho_2(t2_i) = 2\pi f_2(x2_j),$$

$$j = 1,2,\ldots,n; \tag{11}$$

Thus, we have 2n equations with 2n unknowns. It is possible to solve the received system of the linear equations with the help to one of the direct methods, for example, by Gauss modified method.

Numerical Experiments and Results

As it was mentioned above, from the systems of algebraic equations (10) and (11) we defined numerical values of an unknown characteristic function for disclosing the cracks in *n* and 2*n* knot points, respectively. The system of the linear algebraic equations and corresponding graphics were solved and constructed by programming system Matlab.

Now first of all, let us consider approximate solution of the problem (5). The base system of integral equations (5) is solved by a discrete singular method. When the coefficients of the term containing unmoving specifics a_1=a_2=0. On both sides of the body there are equal quality materials b_1=b_2=0, then for the functions $f_1(x)$ and $f_2(x)$ we are studying the following three cases:

a) $f1(x)=1$, $f2(x)=1$. (variant 1),

b) $f1(x)=1$, $f2(x)=2$. (variant 2),

c) $f1(x)=2$, $f2(x)=1$. (variant 3).

At the initial stage the base system of singular integral equations (5) is solved by a discrete singular method in the interval [-1, 1] when the number of the crack splitting were n=10 or n=20.The performed experimental calculations showed: a) on what part of the body the loading was more, consequently,the functions of disclosing of cracks at the ends of the crack grew quickly; b) behavior of the characteristic functions of the crack significantly depended on a ratio of loadings on both sides of the crack. Namely, when these loadings on both sides of the crack were equal, then graphs of characteristic functions of a crack were asymmetrical (skew-symmetric) and when a ratio of loadings on both sides of a crack was equal to two, then there were observed neither symmetry nor anti-symmetry graphs which corresponds to the physical nature

Table 1. The values of coefficients of the intensity of stresses for different number of splitting *n*, at constant and uniform loadings $f1(x)=1,\ f2(x)=1(v.1), f1(x)=1,\ f2(x)=2(v.2), f1(x)=2\ ,\ f2(x)=1(v.3)$.

Var.	n	32	64	128	256	512	1024	2048	4096
1	SIF1	2.464	2.485	2.496	2.501	2.504	2.505	2.506	2.506
	SIF2	-2.464	-2.485	-2.496	-2.501	-2.504	-2.505	-2.506	-2.506
2	SIF1	-1.137	-1.681	-2.231	-2.784	-3.337	-3.890	-4.443	-4.996
	SIF2	-8.528	-9.136	-9.716	-10.287	-10.848	-11.406	-11.961	-12.515
3	SIF1	8.528	9.136	9.716	10.287	10.848	11.406	11.961	12.515
	SIF2	1.137	1.681	2.231	2.784	3.337	3.890	4.443	4.996

of the task, c) increment of the number of knots (as well as usage of algorithm of crack non-uniform splitting) promoted augmentation of the stability diapason of the characteristic function of the crack and it is important to remark that the fast increments of the characteristic function of the crack were observed only at the end points of the crack. It is very interesting to study the behavior of the main characteristic functions of the crack for both very small and very heavy loading but it must be noted that as we are considering linear problem of the elasticity theory the increment or diminution loading will lead to increment or diminution of the values of relevant solutions in linear way.

In the above research the main objective was to study possible distribution of cracks along a body, to investigate behavior of solutions (characteristic functions of stress), to find out the coefficients of intensity of stress (cis_1, cis_2) or stress intensity factor (sif_1, sif_2) in a vicinity at the ends of the cracks and to predict the crack possible penetration into the body: $sif_1 = \lim\limits_{x\to -1}\sqrt{1+x}\,\rho_2(x)$, $sif_2 = \lim\limits_{x\to +1}\sqrt{1-x}\,\rho_1(x)$.

For this purpose the values of the coefficients of intensity of stress (stress intensity factor) $sif_1 \approx \sqrt{1+x2_1}\,\rho_2(x2_1)$, $sif_2 \approx \sqrt{1-x1_n}\,\rho_1(x1_n)$ were calculated by algorithm of uniform splitting of the interval [-1,+1] accompanied with the increment of the number of splitting two times at each step of the calculations (see Table 1).

Table 1 was scheduled and filled by the approximate values of the coefficients of the intensity of tension sif_1 and sif_2 , which were conformable to number n of uniform splitting of the interval [-1,+1]. It is visible from Table 1 that the absolute values, (magnitudes) of the coefficients of the intensity of tension increase with the increase of the number of division of the interval [-1,+1] and are rather close to the real values of the coefficients of the intensity of tension in a vicinity at the end points of the crack.

Performed calculations showed that if the absolute values of the coefficients of intensity of tension are less than 1 (close to critical limit of distribution of a crack) the cracks do not delovelop. But it was close to 1 and cracks developed very slowly (see [6, 7]). Also numerical experiments showed that increment of loading at the ends of the crack causes increment of values of the coefficients of the intensity of tension. As we consider linear tasks of the elasticity theory the increment or diminution loading will lead to proportional increment or diminution of values of relevant solutions. The last fact gives possibility to make the hypothetical forecasts about developments of a crack.

On the second stage let us consider a question of an approximate solution of (8)-(9). For illustration in Table 2 only two typical examples from performed calculations with the different loading on the crack are presented:

a) $C=0,\ F(x)=1$ (variant 4),

b) $C=1,\ F(x)=1$ (variant 5).

Table 2. The values of coefficients of intensity of stresses for different number of splitting *n*, at constant and uniform loadings $C=0,\ F(x)=1(v.4),\ C=1,\ F(x)=1(v.5)$.

Var.	n	32	64	128	256	512	1024	2048	4096
4	SIF1	0.2723	0.2771	0.2800	0.2809	0.2814	0.2818	0.2819	0.2821
	SIF2	-0.2723	-0.2771	-0.2800	-0.2809	-0.2814	-0.2818	-0.2819	-0.2821
5	SIF1	0.8521	0.8491	0.8477	0.8470	0.8466	0.8466	0.8464	0.8463
	SIF2	0.3075	0.2948	0.2884	0.2853	0.2853	0.2829	0.2825	0.2823

In this case too for calculation of the approximate values $sif_2 \approx \sqrt{x_1}\,\rho(x_1)$, $sif_1 \approx \sqrt{1-x_n}\,\rho(x_n)$ of the coefficients of intensity of stresses $sif_2 = \lim\limits_{x\to 0}\sqrt{x}\,\rho(x)$, $sif_1 = \lim\limits_{x\to +1}\sqrt{1-x}\,\rho(x)$ the uniform spliting of the interval [0,1] with an increment of the number of splitting two times at each step was used. As it is visible from Table 2, increment of loading at the ends of the crack causes augmentation of values of the coefficients of the intensity of tension and the last gives possibility to give hypothetical prediction of the cracks penetration into the body.

Conclusions

The cracks can occur on the pipes at different stages of production, repair and service. Natural phenomena, such as earthquakes, landslides and floods, may cause cracks in the gas (petrol) pipelines and tanks. Augmentation of cracks lead to expansion of volume of leakage that causes great ecocatastrophes and economic expenses. Mathematical simulation is a good mean of investigation of cracks' behavior in the pipelines. So the study of the boundary value problems for the composite bodies weakened by cracks has a great practical significance. Equations of the anti-plane elasticity theory for composite bodies weakened by cracks can be used as an initial approximation for the mathematical models investigating cracks problems in concave bodies. From the theoretical and practical point of view the cases when cracks intersect an interface or penetrate the interface at right angle are more interesting. This method allows us to obtain boundary value problems for differential equations with partial derivatives, and so it is of vital importance to create effective approximate methods for solving the problems.

In the present article an anti-plane problem of the elasticity theory for composite (piece-wise homogeneous) orthotropic body weakened by crack intersecting an interface or reaching it in a right angle was reduced to the singular integral equation (when crack reaches to the interface) and the system (pair) of singular integral equations (when crack intersects the interface). Integral equations in both cases contain an immovable singularity, with respect to the unknown characteristic functions of disclosure of cracks. Behavior of solutions in the neighborhood of the crack endpoints was studied by the method of discrete singularity with uniform division of an interval by knots. Corresponding algorithms were constructed and realized for both cases.The results of calculations obtained by the given algorithms were in a good consonance with the theoretical results and so they give possibility to make hypothetical prediction of the cracks penetration into the body.

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