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Mathematics

On One Property of the Wiener Integral and its Statistical Application

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ABSTRACT. For the Wiener integral, one property of inversion is established. This property is used for construction of nonparametric statistical estimation of the unknown logarithmic derivative for distribution random processes, which is observed in Wiener noise.

Key words: *Wiener integral, logarithmic derivative, nonparametric estimator.*

1. Let the sum of independent random processes $\xi_t = \eta_t + w_t$, where w_t is a standard Wiener process, be under observation. Using the observed values (trajectories) $\xi_t^1, \xi_t^2, \ldots, \xi_t^n$ we are to construct an estimate of the unknown logarithmic derivative of the distribution of the process η_t. For random values (when η_t and w_t do not depend on *t*) this problem is considered in [1] in the one-dimensional case. To solve the problem we apply the technique of nonparametric estimation for Wiener integrals.

In the first part of this work we give one inversion theorem for the Wiener integral. This theorem generalizes the well-known result of Rooney [2] for the one-dimensional case. In the Wiener integral theory this result is of independent interest and enables us to construct in the second part of the work finite-dimensional approximations for estimation of the unknown logarithmic derivative in infinite-dimensional space.

We will consider the space of continuous functions $C = C[0,1]$ on $[0,1]$, the σ -algebra of Borel subsets and the Wiener measure μ_w on it. Let further $x = x(t) \in C$ and $t_1, t_2, \ldots, t_n$ be points from $[0,1]$. We will denote by $x^{(n)}$ the broken line with vertices at the points $(t_j, x(t_j))$ $(j = 1, \ldots, n)$. Let $F(x)$ be some functional on *C*. It is assumed that $F(x^{(n)}) = F_n(x_1, \ldots, x_n)$. It is understood that for $n \to \infty$ we have $F(x^{(n)}) \to F(x)$ in the sense of strong convergence.

It is well known ([3]) that for a continuous bounded functional $F(x)$ Lebesgue integral with respect to a Wiener measure (it is this integral that is called the Wiener integral) can be defined by the equality

$$\int_C F(x)\mu_w(dx) = \lim_{n\to\infty} \frac{1}{\sqrt{\pi^n t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})}} \times$$

$$\times \int_{E_n} F_n(x_1,\dots,x_n)\exp\left[-\frac{x_1^2}{t_1}-\sum_{j=1}^{n-1}\frac{(x_{j+1}-x_j)^2}{t_{j+1}-t_j}\right]dx_1\cdots dx_n\,, \tag{1}$$

where E_n is an n-dimensional Euclidean space. An analogous formula also holds for multiple Wiener integrals.

For a continuous bounded functional $f(x)$ on $C[0,1]$ let us consider the transform which we call the Wiener transform

$$g(y)=\int_{C[0,1]} f(y-x)\mu_w(dx) \tag{2}$$

(meaning, in the end, the consideration of the sum of independent random elements, we make at once the following remark. Let E be a separable linear topological space with a Borel σ -algebra of its subsets $\mathrm{B}(E)$. Let further the independent random elements ξ and η with values in E and with probability distributions $\mu(A)=P\{\omega:\xi\in A\}$ and $\nu(A)=P\{\omega:\eta\in A\}$, respectively, on $\mathrm{B}(E)$ be defined on the probability space $\{\Omega,\Im,P\}$. For the sum $\zeta=\xi+\eta$ we can find a distribution $\lambda(A)=P\{\omega:\zeta\in A\}$ in the convolution form

$$\lambda(A)=\int_E \mu(A-t)\nu(dt),\quad A\in\mathrm{B}(E), \tag{*}$$

where $\mu(A-t)$ denotes the shift of μ onto the vector $t\in E$. Indeed, by the generalized formula of total probability we obtain

$$P\{\xi+\eta\in A\}=\int_E P\{\xi\in A-t\mid\eta=t\}P\{\eta\in dt\}=\int_E P\{\xi\in A-t\}\nu(dt).$$

Which is equivalent to (*).

If $\pi(A)$ is a measure such that simultaneously both λ and μ are absolutely continuous with respect to π (for example, if $\pi=0{,}5(\lambda+\mu)$), then (*) can be rewritten in the form

$$\rho_\lambda(x)=\int_E \rho_\mu(x-t)\nu(dt),\quad \text{where}\quad \rho_\lambda(x)=\frac{d\lambda}{d\pi}(x)\quad\text{and}\quad \rho_\mu(x)=\frac{d\mu}{d\pi}(x),$$

which is usually applied in the finite-dimensional case for distribution densities with respect to a Lebesgue measure).

The role which this expression plays in various problems of analysis and in particular in the theory of parabolic differential equations is well-known (see, for example, [4]). We are interested in the question of inversion of transform (2).

For the sake of simplicity we give the result from [2] which has been mentioned above. Let ξ be a standard, normally distributed random value with parameters $(0,\sigma^2)$. Let further $f(x)$, $x\in R$, be a function such that the convolution

$$g(x)=\frac{1}{\sigma}\cdot\varphi\left(\frac{x}{\sigma}\right)*f(x),\quad\text{where}\quad \varphi(x)=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}$$

is defined.

Theorem 1 (Rooney [2]). *Under the above assumptions*

$$f(x)=\lim_{\tau\to1-}\sum_{k=0}^{\infty}(-1)^k g^{(2k)}(x)\frac{(\sigma^2\tau)^k}{2^k k!}.$$

We remark immediately that if the function $g(x)$ and all its derivatives $g^{(m)}(x)$, $m=1,2,\dots$, are bounded when taken as a whole, then, passing in this sum to the limit, we can write

$$f(x)=\sum_{k=0}^{\infty}(-1)^k g^{(2k)}(x)\frac{\sigma^{2k}}{2^k k!}.$$

Such simple variant of Rooney's theorem admits generalization.

Theorem 2. *Let the function* $g(x_1, x_2, \ldots, x_n)$ *have derivatives of all orders and, when taken together with its derivatives as a whole, be bounded. Then the integral equation*

$$g(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - t, \ldots, x_n - t) \exp\left\{-\frac{t^2}{2\sigma^2}\right\} dt \tag{3}$$

has a solution in the class of bounded functions. A solution of (3) *can be represented as*

$$f(x) = \sum_{k=0}^{\infty} (-1)^k d^{2k} g(x) \frac{\sigma^{2k}}{2^k k!},$$

where $x = (x_1, x_2, \ldots, x_n)$ *and* $d^s \varphi = \sum_{\substack{i_1 + \cdots + i_n = s \\ i_k \geq 0,\ k=1,\ldots,n}} \frac{\partial^s \varphi}{\partial^{i_1} x_1 \cdots \partial^{i_s} x_n}$ *is a total derivative of s-th order for the function* $\varphi = \varphi(x_1, \ldots, x_n)$, *where s is a non-negative integer number.*

Proof. The unknown function will be sought formally from under the integral as a Maclaurin series at the point $t = 0$:

$$f(x_1 - t, x_2 - t, \ldots, x_n - t) = \sum_{k=1}^{\infty} \frac{d^{k-1} f}{dt^{k-1}} t^{k-1}.$$

Substituting this expression in (3) and taking into account that for $m > 0$

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} t^m \exp\left\{-\frac{t^2}{2\sigma^2}\right\} dt = \begin{cases} 0 & \text{for } m = 2k-1 \\ \sigma^{2k} 1 \cdot 3 \cdots (2k-1) & \text{for } m = 2k \end{cases}$$

we obtain

$$g(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) + \frac{1}{2!}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\right)^2 \sigma^2 \cdot 1 + \frac{1}{4!}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\right)^4 \sigma^4 \cdot 1 \cdot 3 + \cdots +$$

$$+ \frac{1}{(2k)!}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\right)^{2k} \sigma^{2k} \cdot 1 \cdot 3 \cdots (2k-1) + \cdots.$$

After elementary transformations we can write

$$g = f + \frac{\sigma^2}{2 \cdot 1!} d^2 f + \frac{\sigma^4}{2^2 \cdot 2!} d^4 f + \cdots + \frac{\sigma^{2k}}{2^k \cdot k!} d^{2k} f + \cdots. \tag{4}$$

We differentiate this equality two times and multiply both parts of the resulting equality by the constant $\frac{\sigma^2}{2^1 \cdot 1!}$. After that we differentiate (4) four times and multiply both parts of the resulting equality by the constant $\frac{\sigma^4}{2^2 \cdot 2!}$ and so on. In the general case we differentiate equality (4) $2k$ times and multiply both parts of the resulting equality by $\frac{\sigma^{2k}}{2^k \cdot k!}$ and so on. In addition to (4), we obtain a number of equalities:

$$\frac{\sigma^2}{2^1 \cdot 1!} d^2 g = \frac{\sigma^2}{2^1 \cdot 1!} d^2 f + \frac{\sigma^4}{2 \cdot 1! \cdot 2^1 \cdot 1!} d^4 f + \frac{\sigma^6}{2^2 \cdot 2! \cdot 2^1 \cdot 1!} d^6 f + \cdots + \frac{\sigma^{2k+2}}{2^k \cdot k! \cdot 2^1 \cdot 1!} d^{2k+2} f + \cdots, \tag{5}$$

$$\frac{\sigma^4}{2^2 \cdot 2!} d^4 g = \frac{\sigma^4}{2^2 \cdot 2!} d^4 f + \frac{\sigma^6}{2 \cdot 1! \cdot 2^2 \cdot 2!} d^6 f + \frac{\sigma^8}{2^2 \cdot 2! \cdot 2^2 \cdot 2!} d^8 f + \cdots + \frac{\sigma^{2k+4}}{2^k \cdot k! \cdot 2^2 \cdot 2!} d^{2k+4} f + \cdots \tag{6}$$

....

$$\frac{\sigma^{2k}}{2^k \cdot k!} d^{2k} g = \frac{\sigma^{2k}}{2^k \cdot k!} d^{2k} f + \frac{\sigma^{2k+2}}{2 \cdot 1! \cdot 2^k \cdot k!} d^{2k+2} f + \frac{\sigma^{2k+4}}{2^2 \cdot 2! \cdot 2^k \cdot k!} + \cdots + \frac{\sigma^{4k}}{2^k \cdot k! \cdot 2^k \cdot k!} d^{4k} f + \cdots$$

Now from (4) we subtract (5), add (6), subtract the next equality, add the next equality and so on. We do so, taking into account the combinatorial identity

$$\sum_{k=0}^{m}(-1)^k \frac{1}{(m-k)!\,k!}=0 .$$

Then we finally obtain

$$f(x_1,x_2,\dots,x_n)=g(x_1,x_2,\dots,x_n)-\left(\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}\right)^2\frac{\sigma^2}{2^1\cdot 1!}+\left(\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}\right)^4\frac{\sigma^4}{2^2\cdot 2!}-\cdots \tag{7}$$

which coincides with the equality we wanted to prove. All these formal transformations are justified because, as can be easily verified, by the condition of the theorem the absolute value of the general term of the formal series (7) $d^{2k}g\dfrac{\sigma^{2k}}{2^k\cdot k!}$ vanishes at a sufficiently quick rate and there (7) converges absolutely and uniformly. The theorem is proved.

Corollary 1. *In the conditions of Theorem* 2 *we consider the equality*

$$g(x_1,x_2,\dots,x_n)=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}f(x_1-\lambda_1 t,x_2-\lambda_2 t,\dots,x_n-\lambda_n t)\exp\left\{-\frac{t^2}{2\sigma^2}\right\}dt . \tag{8}$$

Then

$$f(x_1,x_2,\dots,x_n)=\sum_{k=0}^{\infty}(-1)^k\left(\sum_{i=1}^{n}\lambda_i\frac{\partial g}{\partial x_i}\right)^{2k}\frac{\sigma^{2k}}{2^k\cdot k!}. \tag{9}$$

Indeed, if we introduce the notation

$$\varphi(x_1,x_2,\dots,x_n)=g(\lambda_1x_1,\lambda_2x_2,\dots,\lambda_nx_n),\qquad \psi(x_1,x_2,\dots,x_n)=f(\lambda_1x_1,\lambda_2x_2,\dots,\lambda_nx_n),$$

$$\frac{x_k}{\lambda_k}=y_k ,\qquad k=1,2,\dots,n .$$

then (8) takes the form

$$\varphi(y_1,y_2,\dots,y_n)=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\psi(y_1-t,y_2-t,\dots,y_n-t)\exp\left\{-\frac{t^2}{2\sigma^2}\right\}dt,$$

to which Theorem 2 is applicable. After substitution and transformation we obtain (9).

Let us now turn to (2). Instead of (2) we consider the equality

$$g_n(y_1,y_2,\dots,y_n)=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\psi(y_1-t,y_2-t,\dots,y_n-t)\exp\left\{-\frac{t^2}{2\sigma^2}\right\}dt\times$$

$$\times\int_{E_n}f_n(y_1-x_1,\dots,y_n-x_n)\exp\left[-\frac{x_1^2}{t_1}-\sum_{j=1}^{n-1}\frac{(x_{j+1}-x_j)^2}{t_{j+1}-t_j}\right]dx_n\cdots dx_1 . \tag{10}$$

Introducing the notation

$$\frac{x_1}{\sqrt{t_1}}=\frac{u_1}{\sqrt{2}},\dots,\frac{x_{j+1}-x_j}{\sqrt{t_{j+1}-t_j}}=\frac{u_{j+1}}{\sqrt{2}},\dots,\frac{x_n-x_{n-1}}{\sqrt{t_n-t_{n-1}}}=\frac{u_n}{\sqrt{2}},$$

we write (10) in the form

$$g_n(y_1,\dots,y_n)=\frac{1}{\sqrt{(2\pi)^n}}\times$$

$$\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_n\left(y_1-\sqrt{\frac{t_1}{2}}u_1, y_2-\sqrt{\frac{t_1}{2}}u_1-\sqrt{\frac{t_2-t_1}{2}}u_2,\dots,y_n-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_n-t_{n-1}}{2}}u_n\right)\times$$

$$\times\exp\left[-\frac{u_1^2}{2}-\dots-\frac{u_n^2}{2}\right]du_n\cdots du_1 . \tag{11}$$

Let us apply Theorem 2 to equality (11) for $\sigma=1$. For this we rewrite this equality as

$$g_n(y_1,\dots,y_n)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_n\left(y_1-\sqrt{\frac{t_1}{2}}u_1, y_2-\sqrt{\frac{t_1}{2}}u_1-\sqrt{\frac{t_2-t_1}{2}}u_2,\dots,y_{n-1}-\sqrt{\frac{t_1}{2}}u_1-\dots-\right.\right.$$

$$\left.\left.-\sqrt{\frac{t_{n-1}-t_{n-2}}{2}}u_{n-1}, y_n-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_n-t_{n-1}}{2}}u_n\right)\exp\left[-\frac{u_1^2}{2}-\dots-\frac{u_{n-1}^2}{2}\right]du_{n-1}\cdots du_1\right\}e^{-\frac{u_n^2}{2}}du_n .$$

Using Corollary 1, we obtain

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_n\left(y_1-\sqrt{\frac{t_1}{2}}u_1,\dots,y_{n-1}-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_{n-1}-t_{n-2}}{2}}u_{n-1}, y_n-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_{n-1}-t_{n-2}}{2}}u_{n-1}\right)\times$$

$$\times\exp\left[-\frac{u_1^2}{2}-\dots-\frac{u_{n-1}^2}{2}\right]du_{n-1}\cdots du_1=\sum_{k=0}^{\infty}(-1)^k\frac{\partial^{2k}g_n(y_1,y_2,\dots,y_n)}{\partial y_n^{2k}}\frac{(t_n-t_{n-1})^k}{2^{2k}k!}. \tag{12}$$

For further use of Corollary 1, we rewrite (12) in the form

$$\sum_{k_n=0}^{\infty}(-1)^{k_n}\frac{\partial^{2k_n}g_n(y_1,y_2,\dots,y_n)}{\partial y_n^{2k_n}}\frac{(t_n-t_{n-1})^{k_n}}{2^{2k_n}k_n!}=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_n\left(y_1-\sqrt{\frac{t_1}{2}}u_1,\dots,y_{n-2}-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_{n-2}-t_{n-3}}{2}}u_{n-2},\right.\right.$$

$$\left. y_{n-1}-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_{n-1}-t_{n-2}}{2}}u_{n-1}, y_n-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_{n-1}-t_{n-2}}{2}}u_{n-1}\right)\times$$

$$\left.\times\exp\left[-\frac{u_1^2}{2}-\dots-\frac{u_{n-2}^2}{2}\right]du_{n-2}\cdots du_1\right\}e^{-\frac{u_{n-1}^2}{2}}du_{n-1} .$$

By formula (9) we obtain

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_n\left(y_1-\sqrt{\frac{t_1}{2}}u_1,\dots,y_{n-2}-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_{n-2}-t_{n-3}}{2}}u_{n-2}, y_{n-1}-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_{n-2}-t_{n-3}}{2}}u_{n-2}, y_n-\right.$$

$$\left.-\sqrt{\frac{t_1}{2}}u_1-\dots-\sqrt{\frac{t_{n-2}-t_{n-3}}{2}}u_{n-2}\right)\exp\left[-\frac{u_1^2}{2}-\dots-\frac{u_{n-2}^2}{2}\right]du_{n-2}\cdots du_1=$$

$$=\sum_{k_{n-1}=0}^{\infty}\sum_{k_n=0}^{\infty}(-1)^{k_{n-1}+k_n}\left(\frac{\partial^{k_{n-1}}}{\partial y_{n-1}^{k_{n-1}}}+\frac{\partial^{k_{n-1}}}{\partial y_n^{k_{n-1}}}\right)^2\left(\frac{\partial^{k_n}}{\partial y_n^{k_n}}\right)^2 g_n(y_1,\dots,y_{n-1},y_n)\frac{(t_n-t_{n-1})^{k_n}(t_{n-1}-t_{n-2})^{k_{n-1}}}{2^{2(k_{n-1}+k_n)}k_{n-1}!k_n!} .$$

Proceeding so step by step, n-times altogether, we finally obtain

$$f_n(y_1,y_2,\ldots,y_n)=$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\sum_{k_n=0}^{\infty}(-1)^{k_1+k_2+\cdots+k_{n-1}+k_n}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}+\cdots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_1}\left(\frac{\partial}{\partial y_2}+\cdots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_2}\times$$

$$\times\cdots\times\left(\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_{n-1}}\left(\frac{\partial}{\partial y_n}\right)^{2k_n} g_n(y_1,y_2,\ldots,y_{n-1},y_n)\frac{t_1^{k_1}(t_2-t_1)^{k_2}\cdots(t_n-t_{n-1})^{k_n}}{2^{2(k_1+k_2+\cdots+k_n)}k_1!k_2!\cdots k_n!}. \tag{13}$$

Thus we have shown that (10) can be inverted and the inverse is written in form (13). Note that in formula (13) we assume a step-by-step application of differential equations to the functions $g_n(y_1,\ldots,y_n)$. The power of the gradient and the partial gradient is calculated as usual by the Newton formula for higher derivatives.

Let us assume that $g(x)$ is taken from the class of bounded functionals together with their derivatives. Then we can assert that the series in (13) converge uniformly and absolutely.

Indeed, as seen from equality (10), there exist all derivatives $\dfrac{\partial^{2(k_1+\cdots+k_n)}g_n(x_1,\ldots,x_n)}{\partial x_1^{2k_1}\cdots\partial x_n^{2k_m}}$ and they are bounded. Then for the general term in (13) we obtain an estimate

$$\left|\left(\frac{\partial}{\partial y_1}+\cdots+\frac{\partial}{\partial y_n}\right)^{2k_1}\cdots\left(\frac{\partial}{\partial y_n}\right)^{2k_n} g_n(y_1,\ldots,y_n)\frac{t_1^{k_1}\cdots(t_n-t_{n-1})^{k_n}}{2^{2(k_1+\cdots+k_n)}k_1!\cdots k_n!}\right|\le$$

$$\le\frac{M\cdot n^{k_1}(n-1)^{k_2}\cdots 2^{k_{n-1}}1^{k_n}}{2^{2(k_1+\cdots+k_n)}k_1!\cdots k_n!}t_1^{k_1}\cdots(t_n-t_{n-1})^{k_n}\le\frac{C}{2^{k_1+\cdots+k_n}}.$$

(In the expression on the left n is assumed to be the current constant, while $k_1,\ldots,k_n$ are the variables of the general term of the series).

This estimate in (13) gives the desired proof.

Since the series in (13) converge absolutely and uniformly, we can pass to the limit under the summation sign. Then we obtain in the limit the inverse of the Wiener transform (2):

$$f(y)=\lim_{n\to\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\sum_{k_n=0}^{\infty}(-1)^{k_1+k_2+\cdots+k_{n-1}+k_n}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}+\cdots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_1}\times$$

$$\times\left(\frac{\partial}{\partial y_2}+\cdots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_2}\cdots\left(\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_{n-1}}\left(\frac{\partial}{\partial y_n}\right)^{2k_n}\times$$

$$\times g_n(y_1,y_2,\ldots,y_{n-1},y_n)\frac{t_1^{k_1}(t_2-t_1)^{k_2}\cdots(t_n-t_{n-1})^{k_n}}{2^{2(k_1+k_2+\cdots+k_n)}k_1!k_2!\cdots k_n!}. \tag{14}$$

Therefore the following statement is true.

Theorem 3. *The inverse transform for the Wiener transform* (2) *exists in the class of functionals* $g(x)$ *together with their derivatives of all orders which are bounded if taken as a whole and the inverse transform can be calculated by formula* (14) *where* $g_n(x_1,\ldots,x_n)$ *is the orthogonal projection of the functional* $g(x)$.

Remark 1. Frequently, instead of (1) the following definition of a cylindrical Wiener measure is used:

$$\int_C F(x)\mu_w(dx)=$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{\pi^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})}}\int_{E_n}F_n(x_1,\ldots,x_n)\exp\left[-\frac{x_1^2}{2t_1}-\sum_{j=1}^{n-1}\frac{(x_{j+1}-x_j)^2}{2(t_{j+1}-t_j)}\right]dx_1\cdots dx_n.$$

In that case formula (13) has the form

$$f_n(y_1,y_2,\dots,y_n)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\sum_{k_n=0}^{\infty}(-1)^{k_1+k_2+\dots+k_{n-1}+k_n}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}+\dots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_1}\times$$

$$\times\left(\frac{\partial}{\partial y_2}+\dots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_2}\cdots\left(\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_{n-1}}\left(\frac{\partial}{\partial y_n}\right)^{2k_n}\times$$

$$\times g_n(y_1,y_2,\dots,y_{n-1},y_n)\frac{t_1^{k_1}(t_2-t_1)^{k_2}\cdots(t_n-t_{n-1})^{k_n}}{2^{k_1+k_2+\dots+k_n}k_1!k_2!\cdots k_n!} \tag{15}$$

while (14) is written as

$$f(y)=\lim_{n\to\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\sum_{k_n=0}^{\infty}(-1)^{k_1+k_2+\dots+k_{n-1}+k_n}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}+\dots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_1}\times$$

$$\times\left(\frac{\partial}{\partial y_2}+\dots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_2}\cdots\left(\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_{n-1}}\left(\frac{\partial}{\partial y_n}\right)^{2k_n}\times$$

$$\times g_n(y_1,y_2,\dots,y_{n-1},y_n)\frac{t_1^{k_1}(t_2-t_1)^{k_2}\cdots(t_n-t_{n-1})^{k_n}}{2^{k_1+k_2+\dots+k_n}k_1!k_2!\cdots k_n!}. \tag{16}$$

Remark 2. It is interesting to note that (16) (like (14)) can be used to calculate some Wiener integrals. We illustrate this by several examples:

1. Let $g_n(x_1,\dots,x_n)=x_1$. Then from (16) we obtain $f(x)=x_1=x(t_1)$. Therefore

$$\int_{C[0,1]}[\,y(t_1)-x(t_1)]\mu_w dx=y(t_1).$$

Hence we have

$$\int_{C[0,1]}x(t_1)\mu_w(dx)=Ew_{t_1}=0\,.$$

2. Let $g_n(x_1,\dots,x_n)=x_1^2$. Then from (16) we obtain $f(x)=x^2(t_1)-t_1$. Therefore

$$\int_{C[0,1]}\left[y^2(t_1)-2y(t_1)x(t_1)+x^2(t_1)-t_1\right]\mu_w(dx)=y^2(t_1)\,.$$

Hence, taking into account the preceding example, we have

$$\int_{C[0,1]}x^2(t_1)\mu_w(dx)=Ew_{t_1}^2=t_1.$$

3. By analogous calculations, for $g_n(x_1,\dots,x_n)=(x_2-x_1)x_1\quad(t_1<t_2)$ we show that

$$\int_{C[0,1]}x(t_1)\,x(t_2)\,\mu_w(dx)=Ew_{t_1}w_{t_2}=t_1=\min\{t_1,t_2\}.$$

4. If we take $g_n(x_1,\dots,x_n)=e^{\alpha x_1}$, then $f(x)=\exp\left\{\alpha x(t_1)-\frac{1}{2}\alpha^2 t_1\right\}$ and, after carrying out some calculations, we obtain

$$\int_{C[0,1]}e^{-\alpha x(t_1)}\mu_w(dx)=Ee^{-\alpha w(t_1)}=e^{\frac{1}{2}\alpha^2 t_1}.$$

5. Let us calculate $\int_{C[0,1]}\exp\left\{-\alpha\int_0^1 x(t)dt\right\}\mu_w(dx)$. For this we take $g(x_1,\dots,x_n)=\exp\left\{-\frac{\alpha}{n}\sum_{k=1}^n x_k\right\}$, where $x_k=x\left(\frac{k}{n}\right)$. For $n\to\infty$ we have $g(x_1,\dots,x_n)\to\exp\left\{-\alpha\int_0^1 x(t)dt\right\}$. It is easy to calculate that

$$\left(\frac{\partial}{\partial x_1}+\cdots+\frac{\partial}{\partial x_n}\right)^{2k_1}\cdots\frac{\partial^{2k_n}}{\partial x_n^{2k_n}}\exp\left\{-\frac{\alpha}{n}\sum_{k=1}^{n}x_k\right\}=\frac{(n\alpha)^{2k_1}\cdots(2\alpha)^{2k_{n-1}}(1\alpha)^{2k_n}}{n^{2(k_1+\cdots+k_n)}}\exp\left\{-\frac{\alpha}{n}\sum_{k=1}^{n}x_k\right\}.$$

Therefore

$$f(x)=\lim_{n\to\infty}\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}(-1)^{k_1+\cdots+k_n}\frac{\alpha^{2(k_1+\cdots+k_n)}n^{2k_1}(n-1)^{2k_2}\cdots1^{2k_n}}{n^{3(k_1+\cdots+k_n)}2^{k_1+\cdots+k_n}k_1!\cdots k_n!}\exp\left\{-\frac{\alpha}{n}\sum_{k=1}^{n}x_k\right\}=$$

$$=\lim_{n\to\infty}\exp\left\{-\frac{\alpha^2n^2}{2n^3}\right\}\exp\left\{-\frac{\alpha^2(n-1)^2}{2n^3}\right\}\cdots\exp\left\{\frac{\alpha^1\cdot1}{2n^3}\right\}\exp\left\{-\frac{\alpha}{n}\sum_{k=1}^{n}x_k\right\}=$$

$$=\lim_{n\to\infty}\exp\left\{-\alpha^2\frac{n\left(n+\frac{1}{2}\right)(n+1)}{2n^3}-\frac{\alpha}{n}\sum_{k=1}^{n}x_k\right\}=\exp\left\{-\frac{\alpha^2}{2}-\alpha\int_0^1x(t)dt\mu\right\}.$$

Thus

$$\exp\left\{-\alpha\int_0^1x(t)dt\right\}=\int_{C[0,1]}\exp\left\{-\frac{\alpha^2}{2}-\alpha\int_0^1x(t)dt-\alpha\int_0^1y(t)dt\right\}\mu_w(dy).$$

This implies that

$$\int_{C[0,1]}\exp\left\{-\alpha\int_0^1x(t)dt\right\}\mu_w(dx)=\exp\frac{\alpha^2}{2}.$$

2. Let us proceed to the application of the results obtained. Assume that in $C[0,1]$ we have the sum of random elements $X_t=Y_t+W_t$, where W_t is a standard Wiener process and Y_t does not depend on W_t. Let $X_t^1,X_t^2,\ldots,X_t^m$ be independent observations (trajectories) of an element X_t. We are to estimate the logarithmic derivative $l(x,h)$ of the distribution μ_y of the random process Y_t along *h*.

We consider the points $t_1<t_2<\cdots<t_n\in[0,1]$. Let $\lambda=\max\limits_{j=1,\ldots,n-1}(t_{j+1}-t_j)\to0$ as $n\to\infty$. We introduce the notation $X^n=(X_{t_1},X_{t_2},\ldots,X_{t_n})$, $Y^n=(Y_{t_1},Y_{t_2},\ldots,Y_{t_n})$, $W^n=(W_{t_1},W_{t_2},\ldots,W_{t_n})$ and consider the equality $X^n=Y^n+W^n$. This is the equality in the finite-dimensional space R^n and if we assume the existence of distribution densities, we can write

$$p_X(x_1,\ldots,x_n)=$$

$$=\frac{1}{\sqrt{\pi^nt_1(t_2-t_1)\cdots(t_n-t_{n-1})}}\int_{E_n}p_Y(x_1-y_1,\ldots,x_n-y_n)\exp\left\{-\frac{y_1}{t_1}-\sum_{j=1}^{n-1}\frac{(y_{j+1}-y_j)^2}{t_{j+1}-t_j}\right\}dy_n\cdots dy_1.$$

By formula (15) we have

$$p_Y(y_1,y_2,\ldots,y_n)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\sum_{k_n=0}^{\infty}(-1)^{k_1+k_2+\cdots+k_{n-1}+k_n}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}+\cdots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_1}\times$$

$$\times\left(\frac{\partial}{\partial y^2}+\cdots+\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_2}\cdots\left(\frac{\partial}{\partial y_{n-1}}+\frac{\partial}{\partial y_n}\right)^{2k_{n-1}}\left(\frac{\partial}{\partial y_n}\right)^{2k_n}\times$$

$$\times p_X(y_1,y_2,\ldots,y_{n-1},y_n)\frac{t_1^{k_1}(t_2-t_1)^{k_2}\cdots(t_n-t_{n-1})^{k_n}}{2^{2(k_1+k_2+\cdots+k_n)}k_1!k_2!\cdots k_n!}. \qquad (17)$$

For brevity, we introduce the notation

$$(-1)^{k_1+\cdots+k_n}\frac{t_1^{k_1}\cdots(t_n-t_{n-1})^{k_n}}{2^{k_1+\cdots+k_n}k_1!\cdots k_n!}=\varphi(\vec{t},\vec{k}),\qquad \sum_{k_1}^{\infty}\cdots\sum_{k_n}^{\infty}\sim=\sum_{\vec{k}}\sim,$$

$$\left(\frac{\partial}{\partial y_1}+\cdots+\frac{\partial}{\partial y_n}\right)^{2k_1}\cdots\left(\frac{\partial}{\partial y_n}\right)^{2k_n}=D^{\vec{k}},\qquad (y_1,\ldots,y_n)=\vec{y}.$$

Then (17) takes the form

$$p_Y(\vec{y})=\sum_{\vec{k}}D^{\vec{k}}p_X(\vec{y})\varphi(\vec{t},\vec{k}). \tag{18}$$

In the finite-dimensional case the logarithmic derivative for a distribution with density $p(x)$, $x\in R^n$ is calculated by the formula

$$l_n(x,h_n)=\frac{(\text{grad}p(x),h_n)}{p(x)}, \tag{19}$$

where $(\,\cdot,\cdot\,)$ denotes the scalar product of the vectors in R^n, and h_n is the orthogonal projection of the vector h in R^n.

On finding the values of the observed functions at the selected points, we compose the matrix of observations

$$\begin{array}{l} X_{t_1}^1, X_{t_2}^1,\ldots,X_{t_n}^1\\ X_{t_1}^2, X_{t_2}^2,\ldots,X_{t_n}^2\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ X_{t_1}^m, X_{t_2}^m,\ldots,X_{t_n}^m \end{array}. \tag{20}$$

Matrix (20) consists of independent and equally distributed random values. Using this matrix we construct the estimate for the density (actually unknown) $p_X(\vec{y})$ and then estimate $p_Y(\vec{y})$ by formula (18), and the sought logarithmic derivative by formula (19).

For estimating the unknown density $p_X(\vec{y})$ we use the statistic

$$\hat{p}_X(\vec{y})=\frac{\lambda_n^m}{n}\sum_{i=1}^{n}K(\lambda_n(\vec{y}-X_i)),$$

where $K(\vec{y})=\prod_{j=1}^{m}K_j(y_j)$, $y=(y_1,y_2,\ldots,y_m)$ and K_j, $j=1,2,\ldots,m$ are even positive bounded functions with

$$\int_R K_j(x)dx=1.$$

For a further discussion we take equal $K_j(x)\equiv K(x)$, $j=1,2,\ldots,m$. In our case

$$\hat{p}_X(\vec{y})=\frac{\lambda_n}{n}\sum_{i=1}^{n}\prod_{j=1}^{m}K\left(\lambda_n\left(y_i-X_{t_i}^j\right)\right).$$

Then

$$\text{grad}\hat{p}_X(\vec{y})=\frac{\lambda_n}{n}\sum_{i=1}^{n}\text{grad}\prod_{j=1}^{m}K\left(\lambda_n\left(y_i-X_{t_i}^j\right)\right).$$

From (18) we obtain

$$\text{grad}p_Y(\vec{y})h_n=\frac{\lambda_n}{n}\sum_{\vec{k}}D^{\vec{k}}\,\text{grad}p_X(\vec{y})h_n\varphi(\vec{t},\vec{k}).$$

Now for the logarithmic derivative $l_Y(\vec{y},h_n)$ we can write an estimate in the form

$$l_Y(\vec{y},h_n)=\frac{(\text{grad}p_Y(\vec{y}),h_n)}{p_Y(\vec{y})}=\frac{\sum_{\vec{k}}\sum_{i=1}^{n}D^{\vec{k}}\,\text{grad}\prod_{j=1}^{m}K\left(\lambda_n\left(y_i-X_{t_i}^j\right)\right)h_n^i\varphi(\vec{t},\vec{k})}{\sum_{\vec{k}}\sum_{i=1}^{n}D^{\vec{k}}\prod_{j=1}^{m}K\left(\lambda_n\left(y_i-X_{t_i}^j\right)\right)\varphi(\vec{t},\vec{k})}.$$

So, the following statement is true.

Theorem 4. *If in the space* $C[0,1]$ *we have the observation* $X_t^1, X_t^2, \ldots, X_t^m$ *of the realization of the sum of independent random processes* $X_t = Y_t + W_t$, *where* W_t *is a standard Wiener process, then the estimate of the logarithmic derivative of the distribution of the random process* Y_t *is given by the formula*

$$l_Y(\vec{y}, h) = \lim_{m\to\infty} \frac{\lim\limits_{n\to\infty} \sum\limits_{\vec{k}} \sum\limits_{i=1}^{n} D^{\vec{k}}\, grad \prod\limits_{j=1}^{m} K\left(\lambda_n\left(y_i - X_{t_i}^j\right)\right) h_n^i \varphi\left(\vec{t}, \vec{k}\right)}{\lim\limits_{n\to\infty} \sum\limits_{\vec{k}} \sum\limits_{i=1}^{n} D^{\vec{k}} \prod\limits_{j=1}^{m} K\left(\lambda_n\left(y_i - X_{t_i}^j\right)\right) \varphi\left(\vec{t}, \vec{k}\right)},$$

where $K(x)$ *is an even positive smooth density function,* $\lambda_n \to \infty$, $\frac{\lambda_n^2}{n} \to 0$ *as* $n \to \infty$, *the points* $t_1 < t_2 < \cdots < t_n \in [0,1]$ *are chosen so that* $\lambda = \max\limits_{j=1,\ldots,n-1} \left(t_{j+1} - t_j\right) \to 0$ *as* $n \to \infty$.

მათემატიკა

ვინერის ინტეგრალის ერთი თვისებისა და მისი სტატისტიკური გამოყენების შესახებ

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ნაშრომში დადგენილია ვინერის ინტეგრალის შებრუნების ერთი თვისება, რომელიც გამოიყენება უცნობი ლოგარითმული წარმოებულის არაპარამეტრული სტატისტიკური შეფასების ასაგებად ისეთი შემთხვევითი პროცესის განაწილებისათვის, რომელზეც დაკვირვება ხდება ვინერის შეშფოთების პირობებში.

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