საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 3, №1, 2009
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 3, no. 1, 2009

Mathematics

On the Lattice Isomorphisms of 2-Nilpotent W-Power Hall Groups and Lie Algebras

Maia Chabashvili*, Tengiz Bokelavadze**

* *I. Javakhisvili Tbilisi State University*
** *A.Tsereteli State University, Kutaisi*

(Presented by Academy Member H. Inassaridze)

ABSTRACT. The paper deals with lattice isomorphisms of 2-nilpotent Hall W-power groups and Lie algebras. Analogues of the fundamental theorem of projective geometry are proved. A corresponding example is constructed. *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *nilpotent group, nilpotent Lie algebra, lattice isomorphism.*

We investigate the lattice isomorphisms of W-power Hall groups and Lie algebras. The main definitions and notation are standard and can be found in [1-3] for W-power groups and in [4-6] for Lie algebras.

If G is a W-power group over the ring W, then it is obvious that the set of all W-subgroups forms a complete lattice $L(G)$. Analogously, if a Lie algebra L is defined over the ring K, then $\mathcal{L}(L)$ denotes the complete lattice of all subalgebras. W-power groups G and (Lie algebras L and L_1) G_1 over the rings W and W_1 are lattice-isomorphic if there exists an isomorphism

$$f: \mathcal{L}(G) \to \mathcal{L}(G_1) \quad (f: \mathcal{L}(L) \to \mathcal{L}(L_1)).$$

For the completeness of our exposition we give some necessary definitions and notation.

Definition 1. Let X and Y be W-power groups over the rings W_1 and W_2, respectively. We say that the mapping $f: X \to Y$ is a semi-linear isomorphism with respect to the isomorphism $\sigma: W_1 \to W_2$. If the equality

$$f(x_1^{\alpha_1} x_2^{\alpha_2}) = f(x_1)^{\sigma(\alpha_1)} f(x_2)^{\sigma(\alpha_2)}$$

is fulfilled for any $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathrm{W}_1$.

Let us recall the standard notations:

$Z(X)$ is the center of a group X; $N(A)$ is the normalizer of a subset $A \leq X$; $[A, B]$ is the set of all elements (commutators) of the form $[a, b]$, $a \in A$, $b \in B$; $f: L(X) \to L(Y)$ is a lattice isomorphism. For a subset $A \leq X$ we denote by $<A>$ a W-subgroup generated by A.

The image of a W-subgroup $A \subseteq G$ (of a subalgebra $A \subseteq L$) under f is denoted by $f(A)$. We say that a W-group G is determined by the lattice of W-subgroups if the isomorphism f implies the existence of an isomorphism $\sigma: W \to W_1$ such that G and G_1 are semilinearly isomorphic with respect to σ. Analogously, a Lie algebra L is determined by a lattice of its subalgebras if the isomorphism f implies the existence of a semilinear isomorphism $\mu: L \to L_1$ with respect to σ: $K \to K_1$. If the lattice isomorphism f is induced by some semi-linear isomorphism $\mu: G \to G_1$ ($\mu: L \to L_1$), then we say that the fundamental theorem of projective geometry is valid for a W-group G(for Lie algebra L).

Remark that f is induced by a semilinear isomorphism μ if $f(A) = \mu(A)$ for any W-subgroup (subalgebra) A.

For the validity of the fundamental theorem of projective geometry for W-power Hall groups and Lie algebras see [7-9].

We call a lattice $L(0,1 \in L)$ torsion-free if none of the elements L cover 0. A W-power group G is called torsion-free if the condition $x^{\alpha} = 1$, $\alpha \in W$, $x \in G$, implies either $\alpha = 0$ or $x = 1$. We call a W-power group G proper if the lattice $L(G)$ is torsion-free. It is obvious that the lattice $L(G)$ is torsion-free not for every torsion-free group G, i.e. not every torsion-free group is proper. Indeed, if W is a field, then any W-group G over W is torsion-free and thus it is not proper: for any $x \in G$ the lattice $L(<x>)$ has the form $\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}$, i.e. it consists of two elements. An element $x \in G$ is called proper if the lattice $L(<x>)$ is torsion-free, and it is torsion-free if the W-subgroup $<x>$ is torsion-free.

Proposition. A 2-nilpotent W-group X generated by two torsion-free elements x_1, x_2 is a free nilpotent W-group if and only if the W-subgroup $<[x_1, x_2]>$ is torsion-free.

A free 2-nilpotent W-group generated by two elements is denoted by Ω.

In the general case not every lattice isomorphism $f: L(X) \to L(Y)$ is induced by an isomorphism or implies an isomorphism. The following theorem is true.

Theorem 1. *Let X and Y be torsion-free 2-nilpotent W-power groups over the rings W_1 and W_2, respectively. If $X \neq \Omega$, then $W_1 \cong W_2$ and $X \cong Y$.*

Remark 1. If we discard the condition $X \neq \Omega$, then the theorem is true provided that $W_1 \cong W_2$.

In the general theory of groups there is a long-standing problem: is a lattice isomorphism of a torsion-free nilpotent group induced by a group isomorphism? (this is an analogue of the fundamental theorem of projective geometry). This problem is positively solved for torsion-free groups [10] and for proper Lie algebras [7, 8]. In [9], the problem is solved under the restriction that the center of a nilpotent W-group has rank ≥ 2.

Theorem 2 (fundamental theorem of projective geometry for W-power groups). *Let X and Y be W-power groups defined over the principal ideal domains W_1 and W_2, respectively; $f: L(X) \to L(Y)$ be a lattice isomorphism. If X is a proper 2-nilpotent W-power group, then there exist an isomorphism* $\boldsymbol{\sigma}$: *$W_1 \to W_2$ and a σ-semilinear isomorphism $\mu: X \to Y$ such that $\mu(A) = f(A)$ holds for every subgroup $A \subseteq L(X)$*.

Note that X being proper guarantees the fact that the ring W_1 (and therefore W_2) is not a field. Below we give an example showing that the theorem does not hold for the case of a field.

Example 1. Let $\Omega = <a,b>$ be a free 2-nilpotent W-power group generated by two elements a and b. Assume that the principal ring W is a field. Let us consider an automorphism φ of the lattice $L(\Omega)$, $\varphi \in \mathrm{Aut}[L(\Omega)]$, which preserves all 2-generated subgroups and maps the singly generated subgroups arbitrarily but identically with respect to the modulus of the commutant $z = [a,b]$, i.e.

$$\varphi(<x>) = <x \cdot z>.$$

It is easy to see that the lattice automorphism φ is generated by none of the isomorphisms Ω (by none of the semilinear automorphisms).

Remark 2. If μ is the semilinear isomorphism from the Theorem 2, then the mapping μ^{-1} defined by $\mu^{-1}(x) = [\mu(x)]^{-1}$ is a semilinear anti-isomorphism, i.e.

$$\mu(x_1^{\alpha_1} x_2^{\alpha_2}) = \mu(x_2)^{\sigma(\alpha_2)} f(x_1)^{\sigma(\alpha_1)}$$

for any $x_1, x_2 \in X$, $\alpha_1, \alpha_2 \in W$.

Remark 3. Let $\Omega \subset W$ be a group of invertible elements of a ring W. Then for every $\varepsilon \in \Omega$ the mapping $\mu_{\varepsilon} = \mu^{\varepsilon}$ ($\mu^{\varepsilon} = \mu(x)^{\varepsilon}$) is either a semilinear isomorphism or a semi-linear anti-isomorphism with respect to the same σ: $W \to W_1$.

In [7], the Theorem 2 is proved when a 2-nilpotent W-group G. contains a proper non-abelian W-subgroup. There naturally arises the following question: is a lattice isomorphism of .n-nilpotent (n≥3) W-group G induced by a semilinear isomorphism if G contains a proper n-nilpotent subgroup?

Below we give an example of a nilpotent *W*-group of class n≥3, with a proper *n*-nilpotent *W*-subgroup, the lattice isomorphism of which is not induced by a semilinear automorphism.

Example 1. Let G be *W* – power group; *W* be a principal ideal domain which is not a field.

Assume that

$$G = \langle x_1, x_2, ..., x_{n+1}, k_1, k_2, ..., k_{n-2}, n\geq 3\rangle$$

has the defining relations

$$\begin{cases} x_1x_2x_1^{-1} = x_2x_3, \quad x_1x_ix_1^{-1} = x_ix_{i+1}, \ i = 2,3,...,n, \\ x_2x_3x_2^{-1} = x_3k, \quad x_2k_ix_2^{-1} = k_ik_{i+1}, \ i = 1,2,...,n-3, \\ \exp(k_i) = id(p), \ p \in W \ is\ a \text{ prime element}, \ \exp(x_j) = id(1), \ i = 1,...,n-2, \ j = 1,...,n+1. \end{cases}$$

It is easy to see that G has two generators x_1 and x_2 and $G_p = \langle x_1, x_2^p\rangle$ is a proper *n*-nilpotent subgroup. Every element $g \in G$ can be written uniquely in the form

$$g = x_1^{\alpha_1} x_2^{\alpha_2} b, \ b \in [G,G].$$

Let us define the mapping $\varphi: G \to G$ as follows:

$$\varphi(g) = \begin{cases} g, & if \quad \alpha_1\alpha_2 \in id(p), \\ gk_{n-2}^s, & if \quad \alpha_1\alpha_2 \notin id(p), \quad s + \alpha_1 + \alpha_2 \in id(p). \end{cases}$$

It can be shown that φ transforms every *W*-subgroup to a *W*-subgroup, i.e. defines a lattice isomorphism. However, it is obvious that φ is not a semilinear automorphism.

The following theorem is true in the case of Lie algebras.

Theorem 3. *Let L and L_1 be 2-nilpotent Lie algebras over the rings K and K_1, respectively. If $\mathcal{L}(L) \cong \mathcal{L}(L_1)$ and dim L>3, then L and L_1 are semilinearly isomorphic with respect to the isomorphism of $\sigma: K \to K_1$.*

Remark 4. If the algebras L and L_1 are defined over the same ring *K*, then the restrictive condition dim$L > 3$ can be discarded since if dim$L = 3$, then L and L_1 are free nilpotent Lie algebras of class 2 and therefore $L \cong L_1$. The case dim$L = 2$ is trivial.

In connection with the theorem from [7], for Lie algebras there arises an analogous question: is a lattice isomorphism of *n*-nilpotent Lie algebras ($n\geq 3$) induced by a semilinear isomorphism if the algebra contains a proper *n*-nilpotent subalgebra?

We will give an analogous example providing a negative answer to this question.

Example 2. Let a Lie algebra *L* over the principal ideal domain K be defined as follows:

$L = \langle x_1, x_2, ..., x_{n+1}, y_1, y_2, ..., y_{n-2}\rangle$,

$[x_2,x_3] = x_3$, $[x_i,x_1] = x_{i+1}$, $[x_3,x_2] = y_1$, $[y_i,x_2] = y_{i+1}$,

$A\text{nn}(y_i) = \text{id}(p)$, $2 \neq p$, is a prime element *K*,

$A\text{nn}(x_1) = A\text{nn}(x_2) = ... = A\text{nn}(x_i) = 0$, $i = 1,2, ..., n-3$.

It is not difficult to see that *L* really exists, is nilpotent of class $n\geq 3$ and contains a proper subalgebra.

Moreover, $L = \langle x_1\rangle \cup \langle x_2\rangle$ and the subalgebra $L_0 = \langle x_1\rangle \cup \langle px_2\rangle$ is proper and *n*-nilpotent. Every element $l \in L$ has a unique representation of the form

$$L = \alpha_1 x_1 + \alpha_2 x_2 + y, \ y \in [L,L], \ \alpha_1, \alpha_2 \in \text{K}.$$

Let us define the one-to-one mapping as follows

$$f(l) = \begin{cases} l & if \ \alpha_1\alpha_2 \in id(p), \\ \boldsymbol{l} + sy & if \ \alpha_1\alpha_2 \notin id(p), \ s + \alpha_1 + \alpha_2 \in id(p), \ s \neq 0 \end{cases}$$

It can be shown that the mapping *f* defines a lattice isomorphism and that it is induced neither by a semilinear automorphism nor by a semilinear antiautomorphism.

მათემატიკა

ნილპოტენტური კლასის 2 ჰოლის ხარისხოვანი W-ჯგუფების მესერული იზომორფიზმები და ლის ალგებრები

მ. ჭაბაშვილი, თ. ბოკელავაძე

ი.ჯავახიშვილის თბილისის სახელმწიფო უნივერსიტეტი
ა.წერეთლის სახელმწიფო უნივერსიტეტი, ქუთაისი

(წარმოდგენილია აკადემიკოს ხ. ინასარიძის მიერ)

ნაშრომში დამტკიცებულია: თუ ნილპოტენტური კლასის 2, გრეხვის გარეშე ჰოლის ხარისხოვანი ჯგუფები მესერულად იზომორფულნი არიან, მაშინ ისინი ნახევრადწრფივად იზომორფულებია. დამტკიცებულია აგრეთვე პროექციული გეომეტრიის ძირითადი თეორემა სუფთა 2 კლასის ნილპოტენტური W-ჯგუფებისათვის. ანალოგიური თეორემები დამტკიცებულია ლის ალგებრებისათვისაც. მოყვანილია შესაბამისი კონტრ-მაგალითები.

REFERENCES

1. *Ph. Hall* (1968), Mathematics (Periodic of foreign papers), **12**, 1: 3-36.
2. *A. Tavadze* (2008), Sovremennaya matematika i yeyo prilozheniya, **46** (in Russian).
3. *D.W. Barhes and G.E. Wall* (1964), J. Austr. Math. Soc., vol. **4:** 454-459.
4. *Yu.A. Bakhturin* (1985), Tozhdestva v algebrakh Lie, M. (in Russian).
5. *A.G. Gein* (1987), Izv. vuzov. Matematika, **3**: 18-24 (in Russian).
6. *V.R. Varea* (1983), Proc. Royal Soc., Edinburgh, vol. **94**: 9-13.
7. *M. Chabashvili* (2008), Bull. Georgian Acad. Sci., **175**, 4: 44-46.
8. *A. Lashkhi* (2007), DAN RAN, **417**, 3: 313-315 (in Russian).
9. *A. Lashkhi, T. Bokelavadze* (2006), Bull. Georg. Acad. Sci., **173**, 1: 17-18.
10. *L.E. Sadovskii* (1965), Izv. AN SSSR, **20:** 171-208 (in Russian).

Received November, 2008