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Mathematics

On the Hereditary Integrability and Abstract Lebesgue Integral

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ABSTRACT. The notion of hereditary integrability is introduced in this paper. It is demonstrated that every summable function is hereditary integrable or is equivalent to some of such functions. *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words*: measure space, hereditary integrable.*

As is well known [1-5], for construction of the theory of abstract Lebesgue integral it is sufficient to have a measure space (X, S, μ), where S is σ-ring of subsets of non-wide set X, and μ is some non-negative countable additive set function defined on S.

But, it is natural to expect that if S is generated by some ring of subsets of set X and $S \neq R$, then this theory possesses additional interesting properties.

In particular, such situation occurs in case of usual Lebesgue integral in Euclidian spaces R^n and under construction of a theory of product measure of (X_1, S_1, μ_1) and (X_2, S_2, μ_2).

In this paper some questions of theory of integration, namely in such measure spaces, are studied.

Let R be a ring and $S(R)$ be a σ-ring, generated by ring R.

A finite or countable class of pairwise disjoint sets $\{\mathrm{E}_1, \mathrm{E}_2, \ldots\} \subset R$, whose union is equal to set E is called the partition of set E from class R.

Similarly, a finite or countable class of pairwise disjoint sets $\{e_1, e_2, \ldots\} \subset S(R)$, whose union is equal to set E is called the partition of set E from class $S(R)$.

The notion of hereditary integrability is introduced, which is of decisive significance for various generalizations both point function and set function of Fubini's theorem.

It is demonstrated that every summable function is hereditary integrable or is equivalent to some of such functions.

In the sequel we make use of the notations and terms from [2] and [4].

We shall say that a function f belongs to class $[P; X; R]$, if it has the form

$$f = \sum_{k=1}^{n} a_k \chi_{E_k} \left(E_k \in R; k = 1, \ldots, n \right).$$

We shall call the function f elementary, if it has the form

$$f=\sum_{k=1}^{\infty} a_k \chi_{e_k}\left(e_k \in S; k=1,2,\ldots\right).$$

Theorem 1. *let* (X, S, μ) *be a measure space with finite measure, where S is σ-algebra, generated by algebra R. Let f be a finite, measurable function defined on the set* X. *Whatever the number* $\delta>0$ *and* $\varepsilon>0$ *be, there exists such function* $\varphi_{\delta\varepsilon}\in[P;X;R]$ *that*

$$\mu\left(\left\{x:|f(x)-\varphi_{\delta\varepsilon}(x)|\geq\delta\right\}\right)<\varepsilon.$$

Moreover, if $|f(x)|\leq K(x\in X)$, then also $|\varphi_{\delta\varepsilon}(x)|\leq K(x\in X)$.

Corollary 1. *Under the hypotheses of theorem 1 for any number* $\varepsilon>0$ *there exists such function* $\varphi_\varepsilon\in[P;X;R]$ *that*

$$\mu\left(\left\{x:|f(x)-\varphi_{\varepsilon}(x)|\geq\varepsilon\right\}\right)<\varepsilon$$

(and f is integrable)

$$\int_X |f(x)-\varphi_\varepsilon(x)|\,d\mu<\varepsilon.$$

Let R be a ring, $S(R)$ be σ-ring generated by ring R and μ be a many-valued set function, defined on the class $S(R)$.

We shall say that the function μ is hereditary integrable on the set $E\in R$ with respect to class $S(R)$, if there is such number I that for any number $\varepsilon>0$ we find such partition $D_\varepsilon E\subset R$ that for every continuation $\{E_1,E_2,\ldots\}$ of partition $D_\varepsilon E$, both from class R and class $S(R)$, under every choice of values of function μ, the inequality holds

$$\left|\sum_{k=1}^{\infty}\mu(E_k)-I\right|<\varepsilon.$$

From this definition it follows immediately that if the function μ is hereditary integrable on the set E with respect to class $S(R)$, then it is integrable on the set E both with respect to class R and with respect to class $S(R)$ and all its integrals are equal.

An example is easily constructed, which shows that from simultaneous integrability of function μ on the set $E\in R$ both with respect to class R and with respect to class $S(R)$ its hereditary integrability on the set E with respect to class $S(R)$ does not follow. In fact, let R be the ring of all finite union pairwise disjoint half-closed intervals $[a,b)$ from $E=[0, 1)$ and $S(R)$ be a class of all Borel subsets of $[0, 1)$. Let further μ be equal to Lebesgue measure, if $E\in R$ and $\mu(e)=0$, if $e\in S-R$. It is obvious that

$$(R)\int_E \mu(dE)=1,\quad (S(R))\int_E \mu(dE)=0,$$

and the function μ cannot be hereditary integrable on the set E with respect to class $S(R)$.

Let (X, S, μ) be a measure space, where S is an σ-ring, generated by ring R and a function f be defined on the set $E\in R$. We shall say that a function f is hereditary integrable on the set $E\in R$, if hereditary integrable on the set E with respect to class $S(R)$ is a many-valued function $\nu(e)=f(e)\mu(e)\left(e\subset E, e\in S(R)\right)$, where a function $f(e)\left(e\subset E, e\in S(R)\right)$ takes all values $f(x)$, when x runs a set e.

Theorem 2. *Let* (X, S, μ) *be a finite measure space, where S is σ-ring, generated by ring R. Every function f bounded on the set* $E\in R$ *is hereditary integrable or equivalent to some of such functions.*

Corollary 2. *Every elementary function f is hereditary integrable on the set* $E\in R$ *or equivalent to some of such functions.*

Theorem 3. *A finite measurable function is hereditary integrable on the set* $E\in R$ *if and only if for any number* $\varepsilon>0$ *there exists such partition* $D_\varepsilon E=\{E_1,E_2,\ldots\}\subset R$ *of set E that the inequality*

$$\sum_{k=1}^{\infty}\left(M_k - m_k\right)\mu\left(E_k\right) < \varepsilon$$

holds.

Theorem 4. *If a sequence of hereditary integrable functions* $\{f_n\}$ *uniformly converges on the set* $E \in R$ *to function f, then a function f is also hereditary integrable on the set E.*

Theorem 5. *Every function summable on the set* $E \in R$ *is hereditary integrable on E or equivalent to some of such functions.*

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