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Hydrology

Study of the Stability of Finite Difference Schemes to Solve Saint-Venant Equations

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ABSTRACT. The motion of water flows of a variable mass on the water-permeable bed and their expression by a system of respective differential equations, when the system is being solved by numerical methods using the finite-difference schemes, is discussed. The requirement of stability of such schemes is indicated and a respective criterion applying the approaches existing in this field is adopted. *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *numerical methods, stability of difference schemes.*

These equations describe the non-steady motion of a water flow of a variable mass in consequence of an increase (water inflow) or decrease (water outflow) in the mass (rainfall runoff, river floods, surface irrigation, and other hydroecological processes) [1-5].

If we take a unit of water flow width, the mentioned system of dynamic and continuity equations can be reduced to the form [4]:

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + 2c\frac{\partial c}{\partial x} = gi - A_0\frac{u^2}{c^3} + \frac{u}{c^2}gq, \\ 2u\frac{\partial c}{\partial x} + c\frac{\partial u}{\partial x} + 2\frac{\partial c}{\partial t} + g\frac{q}{c} = 0 \ , \end{cases} \tag{1}$$

where $c = \sqrt{gH}, A_0 = n_0^2 g^{\frac{5}{2}}$, H is water depth, g is acceleration of gravity, n_0 – coefficient of roughness, u – average by section speed, i – bed slope ($i \geq 0$), q – water outflow or inflow of discharge on a unit of length and width (l/sec), t – time.

For the derivative of the bed section (river, canal) these equations have a similar form and make no alterations in the discussed method of study of stability of difference schemes.

In the system (1) the sign q corresponds to the water outflow.

The system (1), in consequence of nonlinearity, resists analytical solution. Therefore, there is a set of attempts of numerical solution of different boundary-value problems of this system with the aid of finite-difference methods [1-4]. However, as practice shows, a computational process may only be implemented in the case of stability of finite-difference schemes in operation. Therefore, a theoretical study of the stability of difference schemes, even within the framework of limited capacities of mathematical methods for linear equations, is of great significance.

It would be more convenient if the system (1) is written down in the characteristic form. For that, let us add and subtract by turns the first and second equations of the system (1), following which we shall obtain:

$$\left.\begin{aligned}(u+c)\frac{\partial u}{\partial x}+2(u+c)\frac{\partial c}{\partial x}+\frac{\partial u}{\partial t}+2\frac{\partial c}{\partial t}=gi-A_0\frac{u^2}{c^3}+gq\frac{u-c}{c^2}\\(u-c)\frac{\partial u}{\partial x}+2(c-u)\frac{\partial c}{\partial x}+\frac{\partial u}{\partial t}+2\frac{\partial c}{\partial t}=gi-A_0\frac{u^2}{c^3}+gq\frac{u+c}{c^2}\end{aligned}\right\}. \tag{2}$$

The first equation corresponds to the direct characteristic, the second – to the reverse characteristic. Derivatives will be approximated by means of an implicit difference scheme with central difference

$$\left.\begin{aligned}\frac{\partial u}{\partial x}=\frac{u_{n+1}^{k+1}-u_{n-1}^{k+1}}{2\Delta},\quad \frac{\partial c}{\partial x}=\frac{c_{n+1}^{k+1}-c_{n-1}^{k+1}}{2\Delta},\\ \frac{\partial u}{\partial t}=\frac{u_n^{k+1}-u_n^k}{\tau},\quad \frac{\partial c}{\partial t}=\frac{c_n^{k+1}-c_n^k}{\tau}.\end{aligned}\right\}, \tag{3}$$

where τ is time increment, Δ is length increment, k is the index of time layer, n is the index of point on each time layer. The computational practice corroborates great stability of non-explicit difference schemes [6-9].

Since coefficients and free terms of the system (2) are taken on the top layer (k + 1), the corresponding system of algebraic differential equations for determining u_n^{k+1} and c_n^{k+1} (n = 1,2,3…N) turns out to be nonlinear. This nonlinearity is caused by the circumstance that the coefficients and free terms, dependent on u and c on the top time layer, are unknown. Therefore, to overcome difficulties associated with the nonlinearity of these equations, various methods are proposed, such as iteration, conversion, and linearization [6, 7].

The method of stability analysis and approximation of difference schemes, given below, is valid for all the three mentioned schemes. Since the system (2) is nonlinear, a theoretical study of the difference scheme stability for this system has been carried out in compliance with the recommendations referred to in works [6, 9]. For the nonlinear system (2), a liner system of equations in variations is designed, being consequently investigated by the local harmonic/wave analysis technique. This technique is sometimes also called the freeze-etch method of variable coefficients [6, 8, 9].

The system (2) in the case of approximations (3) can be written down as follows:

$$\left.\begin{aligned}2r(u+c)\left(c_{n+1}^{k+1}-c_{n-1}^{k+1}\right)+r(u+c)\left(u_{n+1}^{k+1}-u_{n-1}^{k+1}\right)+2\left(c_n^{k+1}-c_n^k\right)+\left(u_n^{k+1}-u_n^k\right)-gi\tau+A_0\frac{u^2}{c^3}\tau-gq\frac{u-c}{c^2}\tau=0,\\2r(c-u)\left(c_{n+1}^{k+1}-c_{n-1}^{k+1}\right)+r(u-c)\left(u_{n+1}^{k+1}-u_{n-1}^{k+1}\right)-2\left(c_n^{k+1}-c_n^k\right)+\left(u_n^{k+1}-u_n^k\right)-gi\tau+A_0\frac{u^2}{c^3}\tau-gq\frac{u+c}{c^2}\tau=0.\end{aligned}\right\}, \tag{4}$$

where $r = \tau/2\Delta$.

Instead of u and c in the system (3.76) let us take respectively $u^*+\bar{u}$ and $c^*+\bar{c}$ (where u^* and c^* are the system's solution, $\bar{u}$ and are small first-order quantities), following which the further transformations are carried out: for linearization purposes free terms, are expanded into a Taylor series, holding only the first-order terms; the sum of terms containing u* and c* (the system solution) is made to vanish. After neglecting the second- and higher-order quantities, a system of difference equations in variations is obtained which, after removal of asterisks at u and c, is reduced to the following form:

$$\left.\begin{aligned}&2r(u+c)\left(\bar{c}_{n+1}^{k+1}-\bar{c}_{n-1}^{k+1}\right)+r(u+c)\left(\bar{u}_{n+1}^{k+1}-\bar{u}_{n-1}^{k+1}\right)+2\left(\bar{c}_n^{k+1}-\bar{c}_n^k\right)+\bar{u}_n^{k+1}-\bar{u}_n^k-\\&-3A_0\frac{u^2}{c^4}\tau\,\bar{c}_n^{k+1}+2A_0\frac{u}{c^3}\tau\,\bar{u}_n^{k+1}+gq\tau\left(\frac{2u}{c^3}-\frac{1}{c^2}\right)\bar{c}_n^{k+1}-\frac{gq\tau}{c^2}\bar{u}_n^{k+1}=0,\\&2r(c-u)\left(\bar{c}_{n+1}^{k+1}-\bar{c}_{n-1}^{k+1}\right)+r(u-c)\left(\bar{u}_{n+1}^{k+1}-\bar{u}_{n-1}^{k+1}\right)-2\left(\bar{c}_n^{k+1}-\bar{c}_n^k\right)+\bar{u}_n^{k+1}-\bar{u}_n^k-\\&-3A_0\frac{u^2}{c^4}\tau\,\bar{c}_n^{k+1}+2A_0\frac{u}{c^3}\tau\,\bar{u}_n^{k+1}+gq\tau\left(\frac{2u}{c^3}+\frac{1}{c^2}\right)\bar{c}_n^{k+1}-\frac{gq\tau}{c^2}\bar{u}_n^{k+1}=0\end{aligned}\right\}. \tag{5}$$

These equations are now linear for the first-order quantities $\bar{u}$ and $\bar{c}$. The zeroth-order quantities u and c play here the role of "frozen" coefficients, being further rated as parameters, together with the quantity q.

The system (5) is studied by the local harmonic analysis technique [6, 9]. The quantities $\bar{u}$ and $\bar{c}$ are represented as follows:

$$\bar{u}_n^{k+1} = \lambda^{k+1} e^{imx}, \qquad \bar{c}_n^{k+1} = \xi^{imx}, \tag{6}$$

where $x = n\Delta$, $i = \sqrt{-1}$, m is whole number.

Substituting the values $\overline{u_n^{k+1}}$, $\overline{c_n^{k+1}}$, according to (6), in (5), after transformation we will obtain:

$$\left.\begin{array}{l} \left[D\tau + 2 - 4\sigma(1+a)i\right]\xi^{k+1} + \left[1 + \tau d + 2\sigma(1+a)i\right]\lambda^{k+1} - 2\xi^k - \lambda^k = 0, \\ \left[D_1\tau - 2 + 4\sigma(1+a)i\right]\xi^{k+1} + \left[1 + \tau d + 2\sigma(a-1)i\right]\lambda^{k+1} + 2\xi^k - \lambda^k = 0 \end{array}\right\}, \tag{7}$$

where

$$D = \left[gq(2a-1) - 3A_0 a^2\right]\frac{1}{c^2}, \quad D_1 = \left[gq(2a+1) - 3A_0 a^2\right]\frac{1}{c^2}.$$

$$d = \frac{1}{c^2}(3A_0 a - gq), \qquad \sigma = \frac{\tau c}{2\Delta}\sin mx, \qquad a = \frac{u}{c}.$$

The transfer matrix is expressed as [9]

$$G = - H^{-1} H_0, \tag{8}$$

where H is the matrix composed of coefficients in the case of ξ^{k+1}, λ^{k+1} of the system (7), and H_0 is the matrix composed of coefficients in the case of λ^k, ξ^k.

Thereafter, the following operations are performed: the inverse matrix H^{-1} is computed and multiplied by the matrix H_0; then formed are the characteristic matrix and the respective characteristic equation, the solution of which produces eigenvalues of the transfer matrix.

Analysis of eigenvalues of the transfer matrix is made for two cases, when $\tau \to 0$ and when τ has the finite value. **In the first case**, eigenvalues of the transfer matrix $G(0, m)$, after solution of the characteristic equation, can be expressed as follows:

$$|\lambda| = \frac{1 + 4o^2(1 \mp a)^2}{1 + 8\sigma^2(1+a^2) + 16\sigma^4(1-a^2)^2}\sqrt{1 + 4\sigma^2(1 \pm a)^2}\ . \tag{9}$$

It may be proved analytically that $\lambda < 1$ in the case of any values σ and a, which is the stability requirement. This condition will be sufficient as well, if $\tau \to 0$, but Δ holds the finite value.

Proving. The denominator of the expression (9) can be represented as the square brackets product

$$\left[1 + 4o^2(a \mp 1)^2\right] \cdot \left[1 + 4o^2(1 \pm a)^2\right]. \tag{10}$$

It is easy to make certain that after multiplication of the square brackets the produced expression agrees with the equality denominator (9). Substituting (10) in the expression (9) instead of the denominator, we shall receive:

$$|\lambda| = 1 \Big/ \sqrt{1 + 4o^2(1 \pm a)^2}\ , \tag{11}$$

wherefrom it is obvious that $|\lambda| < 1$ for any values σ and a.

Conclusion. The non-explicit scheme (3) for the numerical solution of the system (1), within the framework of the linear approximation theory, is stable.

In the second case, when τ has the finite value, the sufficient condition of stability is fulfilled, where the elements of the transfer matrix $G(\tau, m)$ are restricted to a definite area and where its eigenvalues, except for, possibly, one value, lie within a unit circle [6, 9].

The analysis made demonstrated that the elements of the matrix G are restricted, given the cases of a practical interest. However, to prove analytically that $|\lambda|<1$ is problematic. In this case, $|\lambda|$ is expressed by a complicated formula, which we do give here but note that λ, in contrast to formula (9), besides a and σ, depends additionally upon two dimensionless groups $h_1 = A_0\tau/c^2$ and $h_2 = gq\tau/c^2$. The condition satisfiability check $|\lambda|<1$ was carried out numerically in wide-ranging change of variables, upon which depends $|\lambda|$. These studies demonstrated that in the case of $h_1>8h_2$, we always have $|\lambda|<1$, i.e. stability occurs in all the cases of practical interest. The produced formula for λ also implies that the stronger the resistance forces and the lesser the bottom permeability, the more stable is the difference scheme.

Computer-aided realization of the numerical method of solution of the system (1) with the application of the implicit scheme (3) has corroborated the obtained theoretical results concerning stability of the scheme. Besides the fact that calculations go smoothly, the obtained results are in close agreement with the experimental data [1-3].

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სასრულსხვაობიანი სქემების გამოკვლევა მდგრადობაზე სენ-ვენანის განტოლებათა ამოსახსნელად

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