საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 3, №1, 2009
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 3, no. 1, 2009

*Mathematics*

# Some Approximate Properties of the Cezàro Means of Order $\alpha \in ]0,1[$ of Trigonometric Fourier Series and its Conjugate

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**ABSTRACT. Some approximate properties of the Cezàro means of trigonometric Fourier series and its conjugate are established.** 

***Key words:*** *trigonometric Fourier series, conjugate trigonometric series, Cezàro means.*

**1.** Assume that $T=[-\pi,\pi]$ and $f: R \to R$ are functions with period $2\pi$, where $R=]-\infty,+\infty[$. If a function $f \in L(T)$, then $\sigma[f]$ and $\bar{\sigma}[f]$ denote respectively a trigonometric Fourier series and its conjugate of *f*, i.e.,

$$\sigma[f](x)=\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx,$$

$$\bar{\sigma}[f](x)=\sum_{k=1}^{\infty} -b_k \cos kx + a_k \sin kx,$$

where

$$a_k \equiv a_k(f)=\frac{1}{\pi}\int_T f(t) \cos kt\, dt,$$

$$b_k \equiv b_k(f)=\frac{1}{\pi}\int_T f(t) \sin kt\, dt.$$

We denote by $\bar{f}$ the conjugate function of $f$, i.e.

$$\bar{f}(x)=-\frac{1}{2\pi}\lim_{\varepsilon\to 0+}\int_{\varepsilon}^{\pi}[f(x+t)-f(x-t)]\operatorname{ctg}\frac{t}{2}\, dt.$$

Introduce the following designations:

$$\varphi(x,t)=f(x+t)+f(x-t)-2f(x),$$
$$\psi(x,t)=f(x+t)-f(x-t),$$

$$\bar{f}_n(x)=-\frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi}\psi(x,t)\operatorname{ctg}\frac{t}{2}\, dt, \qquad n \in N.$$

Denote by $\Phi$ (see [1]) the class of all functions $\omega:[0,\pi]\to R$ with the properties:

1. $\omega$ is continuous on $[0,\pi]$;

2. $\omega$ is non-decreasing;
3. $\omega(0) = 0$;
4. $\omega(t) > 0$, $0 < t \le \pi$.

The symbols $\sigma_n^\alpha(x, f)$ and $t_n^\alpha(x, f)$ denote respectively the Cezàro means of the series $\sigma[f]$ and $\overline{\sigma}[f]$ of order $\alpha$, namely:

$$\sigma_n^\alpha(x, f) = \frac{1}{\pi}\int_0^\pi [f(x+t) + f(x-t)] K_n^\alpha(t)\, dt,$$

$$t_n^\alpha(x, f) = -\frac{1}{\pi}\int_0^\pi \psi(x+t)\tau_n^\alpha(t)\, dt,$$

where

$$K_n^\alpha(t) = \frac{1}{A_n^\alpha}\sum_{k=0}^n A_{n-k}^{\alpha-1} D_k(t),$$

$$\tau_n^\alpha(t) = \frac{1}{A_n^\alpha}\sum_{k=0}^n A_{n-k}^{\alpha-1} \overline{D}_k(t),$$

$$D_k(t) = \frac{1}{2} + \sum_{k=1}^n \cos kt, \qquad \overline{D}_k(t) = \sum_{k=1}^n \sin kt, \qquad |t| \in ]0, \pi],$$

and

$$A_0^\alpha = 1, \quad A_k^\alpha = \frac{(\alpha+1)\cdots(\alpha+k)}{k!}, \qquad k \in N, \quad \alpha > -1.$$

**Remark.** Below the symbols $A(f, x)$ and $A(f, x, \alpha, \eta)$ are positive finite constants depending on the corresponding parameters.

**2.** In this paper we establish some local approximation properties of Cezàro means $\sigma_n^\alpha(x, f)$ and $t_n^\alpha(x, f)$ of order $\alpha \in ]0,1[$. The obtained results generalize Obreshkoff's [2] results.

The following statements are true.

**Theorem 1.** *Let* $f \in L(T)$, $\omega \in \Phi$, $\alpha \in ]0,1[$ *and* $x \in T$. *If*

$$\int_0^t |\varphi(x,s)|\, ds \le A(f, x)\, t\,\omega(t), \quad 0 < t \le \eta \le \pi,$$

*then*

$$\left|\sigma_n^\alpha(x, f) - f(x)\right| \le A(f, x, \alpha, \eta)\frac{1}{n^\alpha}\int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^{1+\alpha}}\, dt.$$

**Proof**. It is known that $A_k^\alpha = \sum_{j=0}^k A_{k-j}^{\alpha-1}$, $A_k^{\alpha-1} = A_k^\alpha - A_{k-1}^\alpha$, and therefore, taking into account the definitions of $D_n(t)$ and $K_n^\alpha(t)$, we conclude that

$$\frac{2}{\pi}\int_0^\pi K_n^\alpha(t)\, dt = 1.$$

Then we can write

$$\sigma_n^\alpha(x, f) - f(x) = \frac{1}{\pi}\int_0^\pi [f(x+t) + f(x-t) - 2f(x)] K_n^\alpha(t) dt = \frac{1}{\pi}\int_0^\pi \varphi(x,t) K_n^\alpha(t) dt =$$

$$= \frac{1}{\pi}\int_0^{\frac{1}{n}} \varphi(x,t) K_n^\alpha(t) dt + \frac{1}{\pi}\int_{\frac{1}{n}}^{\eta} \varphi(x,t) K_n^\alpha(t) dt + \frac{1}{\pi}\int_{\eta}^{\pi} \varphi(x,t) K_n^\alpha(t) dt = \sum_{i=1}^3 \Gamma_i(f, x, \alpha, \eta) \tag{1}$$

Since $\left\|K_n^\alpha\right\|_c \le A(\alpha)n$, by the conditions of the theorem, equality (1) implies

$$\Gamma_1(f,x,\alpha,n) \le A(\alpha)n\int_0^{\frac{1}{n}}\left|\varphi(x,t)\right|dt \le$$

$$\le A(f,x,\alpha)\omega\left(\frac{1}{n}\right) \le A(f,x,\alpha)\frac{1}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\frac{\omega(t)}{t^{1+\alpha}}dt\,. \tag{2}$$

It is known that( see [3]), $A(\alpha) \le k^{-\alpha}A_k^\alpha \le A_1(\alpha)$, $(A_k^\alpha \sim k^\alpha)$.

$$K_n^\alpha(t) = \varphi_n^\alpha(t) + r_n^\alpha(t),$$

where

$$\varphi_n^\alpha(t) = \frac{\sin\left[\left(n+\frac{1}{2}+\frac{\alpha}{2}\right)t - \frac{\alpha\pi}{2}\right]}{A_n^\alpha\left(2\sin\frac{t}{2}\right)^{1+\alpha}},$$

$$\left|r_n^\alpha(t)\right| \le \frac{A(\alpha)}{nt^2}, \quad \frac{1}{n} \le t \le \pi,$$

Therefore from inequality (1) we can write

$$\left|\Gamma_2(f,x,\alpha,\eta)\right| \le \frac{A(\alpha)}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\frac{\left|\varphi(x,t)\right|}{t^{1+\alpha}}dt\,. \tag{3}$$

Applying the formula of partial integration we obtain

$$\frac{1}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\frac{\left|\varphi(x,t)\right|}{t^{1+\alpha}}dt = \frac{1}{n^\alpha}\left[\int_0^t\left|\varphi(x,s)\right|ds\,\frac{1}{t^{1+\alpha}}\right]_{\frac{1}{n}}^{\eta} + \frac{(1+\alpha)}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\left[\int_0^t\left|\varphi(x,s)\right|ds\right]\frac{dt}{t^{2+\alpha}} =$$

$$= \frac{1}{n^\alpha}\left[\int_0^\eta\left|\varphi(x,s)\right|ds\,\frac{1}{\eta^{1+\alpha}} - n^{1+\alpha}\int_0^{\frac{1}{n}}\left|\varphi(x,s)\right|ds\right] + \frac{(1+\alpha)}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\left[\int_0^t\left|\varphi(x,s)\right|ds\right]\frac{dt}{t^{2+\alpha}}\,.$$

Hence, by the conditions of the theorem, we can conclude that

$$\frac{1}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\frac{\left|\varphi(x,t)\right|}{t^{1+\alpha}}dt \le A(f,x)\frac{\omega(\eta)}{\eta^\alpha n^\alpha} + A(f,x)\omega\left(\frac{1}{n}\right) + \frac{A(f,x,\alpha,\eta)}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\frac{\omega(t)}{t^{1+\alpha}}dt \le \frac{A_1(f,x,\alpha,\eta)}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\frac{\omega(t)}{t^{1+\alpha}}dt.$$

Then, by virtue of estimate (3), we can write

$$\left|\Gamma_2(f,x,\alpha,\eta)\right| \le \frac{A(f,x,\alpha,\eta)}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\frac{\omega(t)}{t^{1+\alpha}}dt\,. \tag{4}$$

It can be easily noted that for the expression $\Gamma_3(f,x,\alpha,\eta)$contained in (1) the following estimation is valid:

$$\left|\Gamma_3(f,x,\alpha,\eta)\right| \le \frac{A(f,x,\alpha,\eta)}{n^\alpha}\,. \tag{5}$$

Thus, by relations (1), (2), (4) and (5) we have that

$$\left|\sigma_n^\alpha(x,f) - f(x)\right| \le A(f,x,\alpha,\eta)\frac{1}{n^\alpha}\int_{\frac{1}{n}}^{\eta}\frac{\omega(t)}{t^{1+\alpha}}dt\,.$$

The Theorem is proved.

From the proved theorem it follows that if $\omega(t) = t^{\alpha}, 0 < \alpha < 1$, then

$$\left|\sigma_n^{\alpha}(x,f) - f(x)\right| \le A(f,x,\alpha,\eta)\frac{\ln n}{n^{\alpha}}.$$

This estimation was proved by Obreshkoff [2].

**Theorem 2.** *Assume that* $f \in L(T)$, $\omega \in \Phi$, $\alpha \in ]0,1[$ *and* $x \in T$. *If*

$$\int_0^t |\psi(x,s)|ds \le A(f,x)t\omega(t), \quad 0 < t \le \eta \le \pi,$$

*then*

$$\left|t_n^{\alpha}(x,f) - \bar{f}_n(x)\right| \le A(f,x,\alpha,\eta)\frac{1}{n^{\alpha}}\int_{\frac{1}{n}}^{\eta}\frac{\omega(t)}{t^{1+\alpha}}dt.$$

**Proof.** It is known that

$$t_n^{\alpha}(x,f) = -\frac{1}{\pi}\int_0^{\pi}\psi(x,t)\tau_n^{\alpha}(t)dt,$$

where

$$\tau_n^{\alpha}(t) = \frac{1}{2}ctg\frac{t}{2} + \psi_n^{\alpha}(t) + \gamma_n^{\alpha}(t), \quad |t| \in ]0,\pi]. \tag{6}$$

We have that

$$t_n^{\alpha}(x,f) - \overline{f_n}(x) = -\frac{1}{\pi}\int_0^{\frac{1}{n}}\psi(x,t)\tau_n^{\alpha}(t)dt + \frac{1}{\pi}\int_{\frac{1}{n}}^{\eta}\psi(x,t)\psi_n^{\alpha}(t)dt +$$

$$+\frac{1}{\pi}\int_{\frac{1}{n}}^{\eta}\psi(x,t)\gamma_n^{\alpha}(t)dt + \frac{1}{\pi}\int_{\eta}^{\pi}\psi(x,t)\left[\psi_n^{\alpha}(t) + \gamma_n^{\alpha}(t)\right]dt \equiv \sum_{i=1}^{4}Y_i(f,x,\alpha,\eta). \tag{7}$$

Using the inequality $\left\|\tau_n^{\alpha}\right\|_c \le A(\alpha)n$, by the conditions of the theorem we conclude that

$$\left|Y_1(f,x,\alpha,\eta)\right| \le A(f,x,\alpha)\,\omega\left(\frac{1}{n}\right). \tag{8}$$

For the expressions $\psi_n^{\alpha}(t)$ and $\gamma_n^{\alpha}(t)$ in equality (6) we know that (see [3] )

$$\psi_n^{\alpha}(t) = -\frac{\cos\left[\left(n + \frac{1}{2} + \frac{\alpha}{2}\right)t - \frac{\alpha\pi}{2}\right]}{A_n^{\alpha}\left(2\sin\frac{t}{2}\right)^{1+\alpha}}$$

and

$$\left|\gamma_n^{\alpha}(t)\right| \le \frac{A(\alpha)}{nt^2}, \quad \frac{1}{n} \le t \le \pi.$$

Therefore, by virtue of these relations and taking into account that

$$A(\alpha) \le k^{-\alpha}A_k^{\alpha} \le A_1(\alpha), \quad (A_k^{\alpha} \sim k^{\alpha}),$$

equality (7) implies

$$\left|Y_2(f,x,\alpha,\eta)\right| + \left|Y_3(f,x,\alpha,\eta)\right| \le \frac{A(\alpha)}{n^{\alpha}}\int_{\frac{1}{n}}^{\eta}\frac{|\psi(x,t)|}{t^{1+\alpha}}dt.$$

Hence, applying the formula of partial integration, from the conditions of the Theorem (see [4]) we obtain

$$\left|Y_2(f,x,\alpha,\eta)\right|+\left|Y_3(f,x,\alpha,\eta)\right|\le \frac{A(f,x,\alpha,\eta)}{n^{\alpha}}\int_{\frac{1}{n}}^{\eta}\frac{\omega(t)}{t^{1+\alpha}}dt\,. \tag{9}$$

Analogously, for the expression $Y_4(f,x,\alpha,\eta)$ contained in (7), the following inequality is valid:

$$\left|Y_4(f,x,\alpha,\eta)\right|\le \frac{\mathrm{A}(f,x,\alpha,\eta)}{n^{\alpha}}\,. \tag{10}$$

According to (7), (8), (9) and (10) we obtain:

$$\left|t_n^{\alpha}(x,f)-\overline{f_n}(x)\right|\le \frac{\mathrm{A}(f,x,\alpha,\eta)}{n^{\alpha}}\int_{\frac{1}{n}}^{\eta}\frac{\omega(t)}{t^{1+\alpha}}dt\,.$$

The Theorem is proved.

*მათემატიკა*

# ფურიეს ტრიგონომეტრიული და მისი შეუღლებული მწკრივების $\alpha\in]0,1[$ რიგის ჩეზაროს საშუალოების ზოგიერთი აპროქსიმაციული თვისება

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(წარმოდგენილია აკადემიკოს გ. ხარატიშვილის მიერ)

**სტატიაში დადგენილია ფურიეს ტრიგონომეტრიული მწკრივის და მისი შეუღლებულის $\alpha\in]0,1[$ რიგის ჩეზაროს საშუალოების ზოგიერთი ლოკალური აპროქსიმაციული თვისება.**

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*Received November, 2008*