Some Approximate Properties of the Cezàro Means of Order \( \alpha \in [0,1] \) of Trigonometric Fourier Series and its Conjugate

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1. Assume that \( T = [-\pi, \pi] \) and \( f : R \to R \) are functions with period 2\( \pi \), where \( R = [-\infty, +\infty] \). If a function \( f \in L(T) \), then \( \sigma[f] \) and \( \bar{\sigma}[f] \) denote respectively a trigonometric Fourier series and its conjugate of \( f \), i.e.,

\[
\sigma[f](x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx,
\]

\[
\bar{\sigma}[f](x) = \sum_{k=1}^{\infty} -b_k \cos kx + a_k \sin kx,
\]

where

\[
a_k = a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt,
\]

\[
b_k = b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt.
\]

We denote by \( \bar{f} \) the conjugate function of \( f \), i.e.

\[
\bar{f}(x) = -\frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} [f(x+t) - f(x-t)] \cot \frac{t}{2} \, dt.
\]

Introduce the following designations:

\[
\phi(x,t) = f(x+t) + f(x-t) - 2f(x),
\]

\[
\psi(x,t) = f(x+t) - f(x-t),
\]

\[
\bar{f}_n(x) = -\frac{1}{2\pi} \int_0^{2\pi} \psi(x,t) \cot \frac{t}{2} \, dt, \quad n \in \mathbb{N}.
\]

Denote by \( \Phi \) (see [1]) the class of all functions \( \omega : [0, \pi] \to R \) with the properties:

1. \( \omega \) is continuous on \( [0, \pi] \);
2. \( \omega \) is non-decreasing;
3. \( \omega(0) = 0 \);
4. \( \omega(t) > 0, \ 0 < t \leq \pi \).

The symbols \( \sigma_n^\alpha(x,f) \) and \( \tau_n^\alpha(x,f) \) denote respectively the Cezàro means of the series \( \sigma[f] \) and \( \tau[f] \) of order \( \alpha \), namely:

\[
\sigma_n^\alpha(x,f) = \frac{1}{\pi} \int_0^\pi [f(x+t)+f(x-t)]K_n^\alpha(t)\,dt,
\]

\[
\tau_n^\alpha(x,f) = -\frac{1}{\pi} \int_0^\pi \varphi(x+t)\tau_n^\alpha(t)\,dt,
\]

where

\[
K_n^\alpha(t) = \frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{\alpha-1} D_k(t),
\]

\[
\tau_n^\alpha(t) = \frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{\alpha-1} \bar{D}_k(t),
\]

\[
D_k(t) = \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^n \cos kt, \quad \bar{D}_k(t) = \sum_{k=1}^n \sin kt, \quad |t| \in [0,\pi],
\]

and

\[
A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha+1)\cdots(\alpha+k)}{k!}, \quad k \in \mathbb{N}, \quad \alpha > -1.
\]

**Remark.** Below the symbols \( A(f,x) \) and \( A(f,x,\alpha,\eta) \) are positive finite constants depending on the corresponding parameters.

2. In this paper we establish some local approximation properties of Cezàro means \( \sigma_n^\alpha(x,f) \) and \( \tau_n^\alpha(x,f) \) of order \( \alpha \in [0,1] \). The obtained results generalize Obreshkoff’s [2] results.

The following statements are true.

**Theorem 1.** Let \( f \in L(\Gamma), \ \omega \in \Phi, \ \alpha \in [0,1] \) and \( x \in T \). If

\[
\int_0^\eta |\varphi(x,s)|\,ds \leq A(f,x)\omega(t), \quad 0 < t \leq \eta \leq \pi,
\]

then

\[
|\sigma_n^\alpha(x,f) - f(x)| \leq A(f,x,\alpha,\eta)\frac{1}{n^\alpha} \int_0^\eta \omega(t)\,dt.
\]

**Proof.** It is known that \( A_k^\alpha = \sum_{j=0}^k A_{k-j}^{\alpha-1}, \ A_k^{\alpha-1} = A_k^\alpha - A_{k-1}^\alpha \), and therefore, taking into account the definitions of \( D_n(t) \) and \( K_n^\alpha(t) \), we conclude that

\[
\frac{2}{\pi} \int_0^\pi K_n^\alpha(t)\,dt = 1.
\]

Then we can write

\[
\sigma_n^\alpha(x,f) - f(x) = \frac{1}{\pi} \int_0^\pi [f(x+t)+f(x-t)-2f(x)]K_n^\alpha(t)\,dt = \frac{1}{\pi} \int_0^\pi \varphi(x,t)K_n^\alpha(t)\,dt =
\]

\[
= \frac{1}{\pi} \int_0^\pi \varphi(x,t)K_n^\alpha(t)\,dt + \frac{1}{\pi} \int_0^\pi \varphi(x,t)\tau_n^\alpha(t)\,dt + \frac{1}{\pi} \int_0^\pi \varphi(x,t)K_n^\alpha(t)\,dt = \sum_{j=1}^3 \Gamma_j(f,x,\alpha,\eta)
\]
Since \( \|K_n^\alpha\| \leq A(\alpha)n \), by the conditions of the theorem, equality (1) implies

\[
\Gamma_1(f, x, \alpha, n) \leq A(\alpha)n \int_0^1 \varphi(x, t) dt \leq A(f, x, \alpha) \frac{1}{n^\alpha} \int_0^\eta \omega(t) dt.
\]

(2)

It is known that (see [3]), \( A(\alpha) \leq k^{-\alpha} A_n^\alpha \leq A_1(\alpha), \quad (A_n^\alpha \sim k^{-\alpha}). \)

\[ K_n^\alpha(t) = \varphi_n^\alpha(t) + \epsilon_n^\alpha(t), \]

where

\[
\varphi_n^\alpha(t) = \frac{\sin \left[ \left( \frac{n + 1}{2} + \frac{\alpha}{2} \right) - \frac{\alpha t}{2} \right]}{A_n^\alpha \left( \frac{1}{2} \sin \frac{1}{2} \right)},
\]

\[ |\epsilon_n^\alpha(t)| \leq \frac{A(\alpha)}{nt^\alpha}, \quad 1 \leq t \leq \pi, \]

Therefore from inequality (1) we can write

\[
\|\Gamma_2(f, x, \alpha, \eta)\| \leq A(\alpha) \frac{1}{n^\alpha} \int_0^\eta \frac{\varphi(x, t)}{t^{1+\alpha}} dt.
\]

(3)

Applying the formula of partial integration we obtain

\[
\frac{1}{n^\alpha} \int_0^\eta \left[ \frac{\varphi(x, t)}{t^{1+\alpha}} \right] dt = \frac{1}{n^\alpha} \left[ \int_0^\eta \varphi(x, s) ds \right] \frac{1}{n^\alpha} \left[ \int_0^\eta \varphi(x, s) ds \right] dt + \frac{1 + \alpha}{n^\alpha} \int_0^\eta \varphi(x, s) ds \left[ \frac{1}{n^\alpha} \left[ \int_0^\eta \varphi(x, s) ds \right] \right] dt = \frac{1}{n^\alpha} \left[ \int_0^\eta \varphi(x, s) ds \right] \frac{1}{n^\alpha} \left[ \int_0^\eta \varphi(x, s) ds \right] dt.
\]

Hence, by the conditions of the theorem, we can conclude that

\[
\frac{1}{n^\alpha} \int_0^\eta \left[ \frac{\varphi(x, t)}{t^{1+\alpha}} \right] dt \leq A(f, x) \frac{\varphi(x, \eta)}{\eta^\alpha n^\alpha} + A(f, x) \frac{\omega(t)}{t^{1+\alpha} n^\alpha} + A(\varphi(x, \eta), \eta^\alpha n^\alpha) \frac{\omega(t)}{t^{1+\alpha} n^\alpha} + A(\varphi(x, \eta), \eta^\alpha n^\alpha) \frac{\omega(t)}{n^\alpha} \leq A_1(f, x, \alpha, \eta) \frac{\omega(t)}{n^\alpha} \frac{\omega(t)}{t^{1+\alpha}} dt.
\]

Then, by virtue of estimate (3), we can write

\[
\|\Gamma_2(f, x, \alpha, \eta)\| \leq A(f, x, \alpha, \eta) \frac{\omega(t)}{n^\alpha} \frac{\omega(t)}{t^{1+\alpha}} dt.
\]

(4)

It can be easily noted that for the expression \( \Gamma_3(f, x, \alpha, \eta) \) contained in (1) the following estimation is valid:

\[
\|\Gamma_3(f, x, \alpha, \eta)\| \leq \frac{A(f, x, \alpha, \eta)}{n^\alpha}.
\]

(5)

Thus, by relations (1), (2), (4) and (5) we have that

\[
|\sigma_n^\alpha(x, f) - f(x)| \leq A(f, x, \alpha, \eta) \frac{1}{n^\alpha} \frac{\omega(t)}{t^{1+\alpha}} dt.
\]
The Theorem is proved. From the proved theorem it follows that if \( \omega(t) = t^\alpha, \alpha \in [0, 1) \), then

\[
\left\| \sigma_n^\alpha(x, f) - f(x) \right\| \leq A(f, x, \alpha, \eta) \frac{\ln n}{n^\alpha}.
\]

This estimation was proved by Obreshkoff [2].

**Theorem 2.** Assume that \( f \in L(T), \omega \in \Phi, \alpha \in [0, 1) \), and \( x \in T \). If

\[
\int_0^1 \psi(x, s) ds \leq A(f, x, \alpha, \eta) \frac{1}{n^{1/\alpha}} \int_{1/n}^{\eta} \omega(t) dt,
\]

then

\[
\left\| \sigma_n^\alpha(x, f) - f(x) \right\| \leq A(f, x, \alpha, \eta) \frac{1}{n^{1/\alpha}} \int_{1/n}^{\eta} \omega(t) dt.
\]

**Proof.** It is known that

\[
t_n^\alpha(x, f) = -\frac{\pi}{\eta} \psi(x, t) \sigma_n^\alpha(t) dt,
\]

where

\[
\sigma_n^\alpha(t) = \frac{1}{\pi} \left\{ \int_0^\pi \psi(x, t) \sigma_n^\alpha(t) dt + \int_0^\eta \psi(x, t) \sigma_n^\alpha(t) dt \right\}.
\]

We have that

\[
t_n^\alpha(x, f) - f_n(x) = -\frac{\pi}{\eta} \int_0^\pi \psi(x, t) \sigma_n^\alpha(t) dt + \int_0^\eta \psi(x, t) \sigma_n^\alpha(t) dt + \int_0^\pi \psi(x, t) \gamma_n^\alpha(t) dt + \int_0^\eta \psi(x, t) \left[ \sigma_n^\alpha(t) + \gamma_n^\alpha(t) \right] dt = \sum_{i=1}^4 Y_i(f, x, \alpha, \eta).
\]

Using the inequality \( \left\| \sigma_n^\alpha \right\| \leq A(\alpha)n \), by the conditions of the theorem we conclude that

\[
\left\| Y_i(f, x, \alpha, \eta) \right\| \leq A(f, x, \alpha) \omega(\frac{1}{n}).
\]

For the expressions \( \psi_n^\alpha(t) \) and \( \gamma_n^\alpha(t) \) in equality (6) we know that (see [3])

\[
\psi_n^\alpha(t) = \frac{\cos \left( \left( n + \frac{1}{2} + \frac{\alpha}{2} \right) t - \frac{\alpha \pi}{2} \right)}{A_n^\alpha \left( 2 \sin \frac{t}{2} \right)^{1+\alpha}}
\]

and

\[
\left\| \gamma_n^\alpha(t) \right\| \leq \frac{A(\alpha)}{n^{\alpha}}, \quad \frac{1}{n} \leq t \leq \pi.
\]

Therefore, by virtue of these relations and taking into account that

\[
A(\alpha) \leq k^{-\alpha} A_n^\alpha \leq A_1(\alpha), \quad (A_n^\alpha - k^\alpha),
\]
equality (7) implies

\[
\left\| Y_1(f, x, \alpha, \eta) \right\| + \left\| Y_2(f, x, \alpha, \eta) \right\| \leq \frac{A(\alpha)}{n^{\alpha}} \frac{1}{t^{1+\alpha}} dt.
\]

Hence, applying the formula of partial integration, from the conditions of the Theorem (see [4]) we obtain
\[ |V_2(f, x, \alpha, \eta)| + |V_3(f, x, \alpha, \eta)| \leq \frac{A(f, x, \alpha, \eta)}{n^\alpha} \int_\eta^\alpha \frac{c(t)}{t^{1+\alpha}} dt. \quad (9) \]

Analogously, for the expression \( Y_4(f, x, \alpha, \eta) \) contained in (7), the following inequality is valid:
\[ |Y_4(f, x, \alpha, \eta)| \leq \frac{A(f, x, \alpha, \eta)}{n^\alpha}. \quad (10) \]

According to (7), (8), (9) and (10) we obtain:
\[ \left| f_n^\alpha (x, f) - f^\alpha_n (x) \right| \leq \frac{A(f, x, \alpha, \eta)}{n^\alpha} \int_\eta^\alpha \frac{c(t)}{t^{1+\alpha}} dt. \]

The Theorem is proved.

REFERENCES


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