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Mathematics

An Approximate Algorithm for One Nonlinear Beam Equation

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(Presented by Academy Member R. Bantsuri)

ABSTRACT. An initial boundary value problem is posed for the Kirchhoff integro-differential equation, which describes the dynamic state of a beam. The solution is approximated with respect to spatial and time variables by the Galerkin method and a difference scheme. For solving the system of nonlinear equations obtained by discretization the Jacobi nonlinear iteration process is used. The error of the Galerkin method is estimated. *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words*: nonlinear beam equation, approximate algorithm, error estimate.*

Let us consider the nonlinear differential equation

$$\frac{\partial^2 u}{\partial t^2}(x,t)+\frac{\partial^4 u}{\partial x^4}(x,t)-\left(\alpha+\beta\int_0^L\left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2}(x,t)=0, \qquad (1)$$

$$0<x<L, \quad 0<t\le T,$$

with the initial boundary conditions

$$u(x,0)=u^0(x), \quad \frac{\partial u}{\partial t}(x,0)=u^1(x),$$
$$u(0,t)=u(L,t)=0, \quad \frac{\partial^2 u}{\partial x^2}(0,t)=\frac{\partial^2 u}{\partial x^2}(L,t)=0, \qquad (2)$$
$$0\le x\le L, \quad 0\le t\le T,$$

where α,β,L and T are some positive constants, $u^0(x)$ and $u^1(x)$ are the given functions, and $u(x,t)$ is the function we want to define. Equation (1) describes the oscillation of a beam in the light of the Kirchhoff theory [1, 2]. Several authors dedicated their works to the study of this equation. Among them are Ball [3, 4], Biler [5], Brito [6, 7], Pereira [8] and Medeiros [9]. Here we consider one of the questions of construction and investigation of approximate algorithms for equation (1). It should be noted that in this direction there are many unsolved problems interesting for investigation. We have used the approach realized by us in [10, 11] for the Kirchhoff string equation.

Assume that the initial functions are represented in the form

$$u^0(x)=\sum_{i=1}^{\infty}a_i^{(0)}\sin\frac{i\pi}{L}x, \quad u^1(x)=\sum_{i=1}^{\infty}a_i^{(1)}\sin\frac{i\pi}{L}x, \qquad 0\le x\le L, \qquad (3)$$

and

$$a_i^{(0)^2} \le \frac{\omega_0}{i^{p+4}}, \quad a_i^{(1)^2} \le \frac{\omega_1}{i^{p}}, \qquad i = 1, 2, \dots, \tag{4}$$

where $p,\ \omega_0,\ \omega_1$ are some positive numbers and also $p > 1$.

Suppose that there exists a solution of problem (1), (2) which is represented in the form

$$u(x,t) = \sum_{i=1}^{\infty} u_i(t) \sin \frac{i\pi}{L} x, \tag{5}$$

where the coefficients $u_i\,(t)$ satisfy the following infinite system of differential equations

$$u_i''(t) + \left(\frac{\pi i}{L}\right)^4 u_i(t) + \left(\frac{\pi i}{L}\right)^2 \left(\alpha + \beta \frac{\pi^2}{2L} \sum_{i=1}^{\infty} j^2 u_j^2(t)\right) u_i(t) = 0, \tag{6}$$

$$i = 1, 2, \dots, \quad 0 < t \le T,$$

with the initial conditions

$$u_i(0) = a_i^{(0)}, \quad u_i'(0) = a_i^{(1)}, \qquad i = 1, 2, \dots \ . \tag{7}$$

Assume also that

$$\text{the series } \sum_{i=1}^{\infty} u_i'^2(t) \quad \text{and} \quad \sum_{i=1}^{\infty} i^4 u_i^2(t) \quad \text{converge.} \tag{8}$$

Let us perform approximation of the solution with respect to the variable x. For this we use the Galerkin method. A solution will be sought in the form of a finite series

$$u_n(x,t) = \sum_{i=1}^{n} u_{ni}(t) \sin \frac{i\pi}{L} x, \tag{9}$$

where the coefficients $u_{ni}(t)$ are a solution of the system of differential equations

$$u_{ni}''(t) + \left(\frac{\pi i}{L}\right)^4 u_{ni}(t) + \left(\frac{\pi i}{L}\right)^2 \left(\alpha + \beta \frac{\pi^2}{2L} \sum_{j=1}^{n} j^2 u_{nj}^2(t)\right) u_{ni}(t) = 0, \tag{10}$$

$$i = 1, 2, \dots, n, \quad 0 < t \le T,$$

with the initial conditions

$$u_{ni}(0) = a_i^{(0)}, \qquad u_{ni}'(0) = a_i^{(1)}, \qquad i = 1, 2, \dots, n. \tag{11}$$

To solve the Cauchy problem (10), (11), let us introduce on the time segment $[0,T]$ a grid with pitch $\tau = T/M$ and nodes $t_m = m\tau,\ m = 0,1,\dots,M$. An approximate value of $u_{ni}(t_m)$ denoted by u_{ni}^m is determined by a Crank-Nicolson type scheme

$$u_{ni\bar{t}t}^{m-1} + \left(\frac{\pi i}{L}\right)^4 \frac{u_{ni}^m + u_{ni}^{m-2}}{2} + \left(\frac{\pi i}{L}\right)^2 \left(\alpha + \beta \frac{\pi^2}{2L} \sum_{j=1}^{n} j^2 \frac{(u_{nj}^m)^2 + (u_{nj}^{m-2})^2}{2}\right) \frac{u_{ni}^m + u_{ni}^{m-2}}{2} = 0, \tag{12}$$

$$i = 1, 2, \dots, n, \quad m = 2, 3, \dots, M,$$

with the conditions

$$\begin{aligned} u_{ni}^0 &= a_i^{(0)}, \\ u_{ni}^1 &= a_i^{(0)} + \tau\, a_i^{(1)} - \frac{\tau^2}{2}\left[\left(\frac{\pi i}{L}\right)^4 + \left(\frac{\pi i}{L}\right)^2 \left(\alpha + \beta \frac{\pi^2}{2L} \sum_{j=1}^{n} j^2 a_j^{(0)2}\right)\right] a_i^{(0)}. \end{aligned} \tag{13}$$

The last part of the algorithm aims at solving the system of nonlinear equations (12), (13). If the counting is performed from layer to layer, then, knowing the results for the preceding layers, at the mth time layer,

$m=2,3,\dots,M$, i.e. for $t=t_m$, we have to solve a system of nonlinear equations with respect to u_{ni}^m, $i=1,2,\dots,n$, which has the form

$$\left[1+\frac{\tau^2}{2}\left(\frac{\pi i}{L}\right)^2\left(\alpha+\left(\frac{\pi i}{L}\right)^2+\beta\frac{\pi^2}{4L}\sum_{j=1}^{n}j^2\left((u_{nj}^m)^2+(u_{nj}^{m-2})^2\right)\right)\right]\left(u_{ni}^m+u_{ni}^{m-2}\right)=2u_{ni}^{m-1}, \tag{14}$$

$$i=1,2,\dots,n.$$

System (14) is solved by the iteration method consisting in calculating successive approximations by Jacobi's rule

$$\left\{1+\frac{\tau^2}{2}\left(\frac{\pi i}{L}\right)^2\left[\alpha+\left(\frac{\pi i}{L}\right)^2+\beta\frac{\pi^2}{4L}\left(i^2\left((u_{ni,k+1}^m)^2+(u_{ni}^{m-2})^2\right)+\sum_{\substack{j=1\\j\neq i}}^{n}j^2\left((u_{nj,k}^m)^2+(u_{nj}^{m-2})^2\right)\right)\right]\right\}\times$$

$$\times\left(u_{ni,k+1}^m+u_{ni}^{m-2}\right)=2u_{ni}^{m-1},\qquad i=1,2,\dots,n,\qquad k=0,1,\dots, \tag{15}$$

where $u_{ni,l}^m$ is the lth iteration approximation of u_{ni}^m, $l=0,1,\dots$. For simplicity we do not take into account the errors in u_{ni}^{m-l}, $l=1,2$, $i=1,2,\dots,n$.

For fixed i (15) is a cubic equation with respect to $u_{ni,k+1}^m$. The Cardano formula [12] allows us to determine $u_{ni,k+1}^m$ in an explicit form. We get

$$iu_{ni,k+1}^m=-\frac{iu_{ni}^{m-2}}{3}-\sum_{l=0}^{1}\left[\frac{s_i}{2}+(-1)^l\left(\frac{s_i^2}{4}+\frac{r_i^3}{27}\right)^{1/2}\right]^{1/3},\quad k=0,1,\dots,\quad i=1,2,\dots n, \tag{16}$$

where

$$r_i=q_i+\frac{2}{3}\left(iu_{ni}^{m-2}\right)^2+\frac{1}{\tau^2 i^2\pi\beta}\left(\frac{2L}{\pi}\right)^3,$$

$$s_i=\frac{2}{3}iu_{ni}^{m-2}\left(q_i+\frac{10}{9}\left(iu_{ni}^{m-2}\right)^2\right)-\frac{1}{\tau^2 i^2\pi\beta}\left(\frac{2L}{\pi}\right)^3\left(\frac{iu_{ni}^{m-2}}{3}+2iu_{ni}^{m-1}\right),$$

$$q_i=\frac{4L}{\pi^2\beta}\left(\alpha+\left(\frac{\pi i}{L}\right)^2\right)+\sum_{\substack{j=1\\j\neq i}}^{n}j^2\left(\left(u_{nj,k}^m\right)^2+\left(u_{nj}^{m-2}\right)^2\right).$$

Thus the proposed algorithm is reduced to the calculation by formula (16).

Now our aim is to estimate the error of the Galerkin method. To achieve this aim it is necessary to introduce several notions and to prove some auxiliary statements. Let λ and μ be n-dimensional vectors, $\lambda=\left(\lambda_i\right)_{i=1}^n$, $\mu=\left(\mu_i\right)_{i=1}^n$. In the first place, we define respectively the scalar product and the norm

$$(\lambda,\mu)_n=\sum_{i=1}^{n}\lambda_i\mu_i,\quad \|\lambda\|_n=(\lambda,\lambda)_n^{\frac{1}{2}}. \tag{17.1}$$

Next, using the functions $u_{ni}(t)$ and the coefficients $a_i^{(l)}$, $i=1,2,\dots,n$, $l=0,1,$ from (9) and (3) we form the vectors

$$\boldsymbol{u}_n(t)=(u_{ni}(t))_{i=1}^n,\quad \boldsymbol{\alpha}_n^l=(a_i^{(l)})_{i=1}^n,\quad l=0,1. \tag{17.2}$$

We also define the matrix and the energetic norm

$$Q_n=\frac{\pi}{L}diag(1,2,\dots,n),\qquad \|\lambda\|_{Q_n^{2l}}=\left(Q_n^{2l}\lambda,\lambda\right)_n^{\frac{1}{2}},\qquad l=1,2. \tag{17.3}$$

Using this notation, (10), (11) can be written in the vector form

$$\boldsymbol{u}_n''(t)+Q_n^4\boldsymbol{u}_n(t)+\left(\alpha+\beta\frac{L}{2}\|\boldsymbol{u}_n(t)\|_{Q_n^2}^2\right)Q_n^2\boldsymbol{u}_n(t)=0,\quad 0<t\le T, \tag{18}$$

$$\boldsymbol{u}_n(0)=\boldsymbol{\alpha}_n^0,\qquad \boldsymbol{u}_n'(0)=\boldsymbol{\alpha}_n^1. \tag{19}$$

By the coefficients of decomposition (5) we form the vector

$$p_n\boldsymbol{u}(t)=\left(u_i(t)\right)_{i=1}^n. \tag{20}$$

Now let us derive an equation for defining the error of the Galerkin method. By the error of the method we understand the difference between the vectors $\boldsymbol{u}_n(t)$ and $p_n\boldsymbol{u}(t)$

$$\Delta\boldsymbol{u}_n(t)=\boldsymbol{u}_n(t)-p_n\boldsymbol{u}(t). \tag{21}$$

Using (20) and (17), the first *n* equations of system (6) and the first *n* equalities from each of the initial conditions (7) are written in the form

$$\left(p_n\boldsymbol{u}(t)\right)''+Q_n^4p_n\boldsymbol{u}(t)+\left(\alpha+\beta\frac{L}{2}\|p_n\boldsymbol{u}(t)\|_{Q_n^2}^2\right)Q_n^2p_n\boldsymbol{u}(t)+\boldsymbol{z}_n(t)=0, \tag{22}$$

$$0<t\le T,$$

$$p_n\boldsymbol{u}(0)=\boldsymbol{\alpha}_n^0,\qquad (p_n\boldsymbol{u})'(0)=\boldsymbol{\alpha}_n^1, \tag{23}$$

where $\boldsymbol{z}_n(t)$ is the vector defined by the formula

$$\boldsymbol{z}_n(t)=\beta\frac{\pi^2}{2L}\left(\sum_{i=n+1}^{\infty}i^2u_i^2(t)\right)Q_n^2p_n\boldsymbol{u}(t). \tag{24}$$

Subtracting (22) and (23) from (18) and (19), respectively, and taking into account (21), we write the equation for the error

$$\left(\Delta\boldsymbol{u}_n(t)\right)''+Q_n^4\Delta\boldsymbol{u}_n(t)+\left(\alpha+\beta\frac{L}{2}\|\boldsymbol{u}_n(t)\|_{Q_n^2}^2\right)Q_n^2\Delta\boldsymbol{u}_n(t)-$$
$$-\beta\frac{L}{2}\left(\|p_n\boldsymbol{u}(t)\|_{Q_n^2}^2-\|\boldsymbol{u}_n(t)\|_{Q_n^2}^2\right)Q_n^2p_n\boldsymbol{u}(t)=\boldsymbol{z}_n(t) \tag{25}$$

with the boundary conditions

$$\Delta\boldsymbol{u}_n(0)=0,\qquad (\Delta\boldsymbol{u}_n)'(0)=0. \tag{26}$$

Equation (25) and conditions (26) are the starting point of the investigation of the problem of method accuracy estimation.

Lemma 1. *The estimate*

$$\|p_n\boldsymbol{u}(t)\|_{Q_n^{2l}}^2\le c_{l-1},\qquad l=1,2, \tag{27}$$

where c_0 *and* c_1 *do not depend on n and t, is valid.*

Proof. We multiply (6) by $2u_i'(t)$ and sum the obtained expression over $i=1,2,\ldots$. If we use (8) and denote

$$\Phi(t)=\sum_{i=1}^{\infty}u_i'^2(t)+\left(\frac{\pi}{L}\right)^4\sum_{i=1}^{\infty}i^4u_i^2(t)+\frac{1}{\beta L}\left(\alpha+\beta\frac{\pi^2}{2L}\sum_{i=1}^{\infty}i^2u_i^2(t)\right)^2, \tag{28}$$

then the result is written as $\Phi'(t)=0$, which means that for $0<t\le T$, we have

$$\Phi(t)=\Phi(0). \tag{29}$$

Using (17), (20) and (28) in (29) we find

$$\|(p_n\boldsymbol{u}(t))'\|_n^2+\|p_n\boldsymbol{u}(t)\|_{Q_n^4}^2+\frac{1}{\beta L}\left(\alpha+\frac{1}{2}\beta L\|p_n\boldsymbol{u}(t)\|_{Q_n^2}^2\right)^2\le\Phi(0). \tag{30}$$

Let us calculate $\Phi(0)$. Applying (28), (7) and (3) we have

$$\Phi(0)=\frac{2}{L}\int_0^L\left[\left(u^1(x)\right)^2+\left(u^{0''}(x)\right)^2\right]dx+\frac{1}{\beta L}\left(\alpha+\beta\int_0^L\left(u^{0'}(x)\right)^2dx\right)^2. \tag{31}$$

From (30), first taking into account that by virtue of (17) $\| p_n\boldsymbol{u}(t)\|_{Q_n^4}\geq\frac{\pi}{L}\| p_n\boldsymbol{u}(t)\|_{Q_n^2}$ we obtain (27) for $l=1$, where

$$c_0=2\frac{1}{\beta L}\left[\left(\left(\frac{\pi}{L}\right)^4+2\alpha\left(\frac{\pi}{L}\right)^2+\beta L\Phi(0)\right)^{\frac{1}{2}}-\left(\left(\frac{\pi}{L}\right)^2+\alpha\right)\right], \tag{32}$$

and then verify the fulfillment of (27) for $l=2$, where

$$c_1=\Phi(0).$$

□

Lemma 2. *The inequality*

$$\|\boldsymbol{u}_n(t)\|^2_{Q_n^2}\leq c_2, \tag{33}$$

where the value c_2 does not depend on t, is fulfilled.

Proof. Multiplying (18) scalarly by $2\boldsymbol{u}_n'(t)$, we obtain $\Phi_n'(t)=0$, where

$$\Phi_n(t)=\|\boldsymbol{u}_n'(t)\|_n^2+\|\boldsymbol{u}_n(t)\|^2_{Q_n^4}+\frac{1}{\beta L}\left(\alpha+\frac{1}{2}\beta L\|\boldsymbol{u}_n(t)\|^2_{Q_n^2}\right)^2. \tag{34}$$

Therefore we get the equality

$$\Phi_n(t)=\Phi_n(0), \tag{35}$$

which together with (34) and (17) imply the fulfillment of (33) where

$$c_2=2\frac{1}{\beta L}\left[\left(\left(\frac{\pi}{L}\right)^4+2\alpha\left(\frac{\pi}{L}\right)^2+\beta L\Phi_n(0)\right)^{\frac{1}{2}}-\left(\left(\frac{\pi}{L}\right)^2+\alpha\right)\right], \tag{36}$$

□

If it is required to calculate or estimate c_2, we must take into account the following relation for $\Phi_n(0)$

$$\Phi_n(0)=\sum_{i=1}^n a_i^{(1)2}+\left(\frac{\pi}{L}\right)^4\sum_{i=1}^n i^4a_i^{(0)2}+\frac{1}{\beta L}\left(\alpha+\frac{1}{2}\beta\frac{\pi^2}{L}\sum_{i=1}^n i^2a_i^{(0)2}\right)^2\leq\Phi(0), \tag{37}$$

which is a result of the application of (34), (19), (17) together with (3) and (31).

□

Comparing (32) and (36) and applying (37), we observe that

$$c_2\leq c_0. \tag{38}$$

Lemma 3. *The inequality*

$$\|\boldsymbol{z}_n(t)\|_n\leq\frac{c_3}{n^{p-1}}, \tag{39}$$

where the value c_3 does not depend on t, is true.

Proof. From (24) and (17) it follows that

$$\| \boldsymbol{z}_n(t) \|_n = \beta \frac{\pi^2}{2L} \sum_{i=n+1}^{\infty} i^2 u_i^2(t) \| p_n \boldsymbol{u}(t) \|_{Q_n^4} . \tag{40}$$

Using (8), let us introduce into consideration the function

$$\Psi_n(t) = \sum_{i=n+1}^{\infty} u_i'^2(t) + \left(\frac{\pi}{L}\right)^4 \sum_{i=n+1}^{\infty} i^4 u_i^2(t) + \left(\frac{\pi}{L}\right)^2 \left(\alpha + \beta \frac{\pi^2}{2L} \sum_{j=1}^{\infty} j^2 u_j^2(t) \right) \sum_{i=n+1}^{\infty} i^2 u_i^2(t). \tag{41}$$

We need to estimate its value for t=0. This estimate is obtained by using (7), (4), (3) and the integral test for the convergence of series. As a result we have

$$\Psi_n(0) = \sum_{i=n+1}^{\infty} a_i^{(1)^2} + \left(\frac{\pi}{L}\right)^4 \sum_{i=n+1}^{\infty} i^4 a_i^{(0)^2} + \left(\frac{\pi}{L}\right)^2 \left(\alpha + \beta \frac{\pi^2}{2L} \sum_{j=1}^{\infty} j^2 a_i^{(0)^2} \right) \sum_{i=n+1}^{\infty} i^2 a_i^{(0)^2} \le$$

$$\le \left(\omega_1 + \omega_0 \left(\frac{\pi}{L}\right)^4 \right) \sum_{i=n+1}^{\infty} \frac{1}{i^p} + \omega_0 \left(\frac{\pi}{L}\right)^2 \left(\alpha + \beta \int_0^L \left(u^{0\prime}(x)\right)^2 dx \right) \sum_{i=n+1}^{\infty} \frac{1}{i^{p+2}} \le$$

$$\le \frac{1}{(p-1)n^{p-1}} \left[\omega_1 + \omega_0 \left(\frac{\pi}{L}\right)^4 + \omega_0 \frac{1}{n^2} \left(\frac{\pi}{L}\right)^2 (p-1)(p+1)^{-1} \left(\alpha + \beta \int_0^L \left(u^{0\prime}(x)\right)^2 dx \right) \right]. \tag{42}$$

Further, comparing the sum $\sum_{i=n+1}^{\infty} i^2 u_i^2(t)$ from (40) with the function $\Psi_n(t)$ from (41), we conclude that

$$\sum_{i=n+1}^{\infty} i^2 u_i^2(t) \le \left(\frac{L}{\pi}\right)^2 \left[\alpha + \left(\frac{\pi}{L}\right)^2 \right]^{-1} \Psi_n(t). \tag{43}$$

Now let us estimate the function $\Psi_n(t)$. After multiplying (6) by $2u_i'(t)$ and summing the resulting equality over $i = n+1, n+2, \dots,$ we obtain

$$\Psi_n'(t) = \beta\pi \left(\frac{\pi}{L}\right)^3 \sum_{j=1}^{\infty} j^2 u_j(t) u_j'(t) \sum_{i=n+1}^{\infty} i^2 u_i^2(t). \tag{44}$$

By (28) and (29) we have

$$\left| \sum_{j=1}^{n} j^2 u_j(t) u_j'(t) \right| \le \frac{1}{2} \left(\frac{L}{\pi}\right)^2 \left(\sum_{j=1}^{n} u_j'^2(t) + \left(\frac{\pi}{L}\right)^4 \sum_{j=1}^{n} j^4 u_j^2(t) \right) \le \tag{45}$$

$$\le \frac{1}{2} \left(\frac{L}{\pi}\right)^2 \Phi(t) = \frac{1}{2} \left(\frac{L}{\pi}\right)^2 \Phi(0).$$

By virtue of (43) – (45), (41) and the Gronwall inequality

$$\Psi_n(t) \le \Psi_n(0) \exp\left[\frac{1}{2} \beta L \Phi(0) \left(\alpha + \left(\frac{\pi}{L}\right)^2 \right)^{-1} t \right]. \tag{46}$$

Applying to (40) inequalities (43), (46), (42) and (27) successively, we come to the conclusion that (39) is fulfilled and also that

$$c_3 = \frac{\beta L c_1}{2(p-1)} \left(\alpha + \left(\frac{\pi}{L}\right)^2 \right)^{-1} \left[\omega_1 + \omega_0 \left(\frac{\pi}{L}\right)^4 + \omega_0 \frac{1}{n^2} \left(\frac{\pi}{L}\right)^2 (p-1)(p+1)^{-1} \times \right.$$

$$\left. \times \left(\alpha + \beta \int_0^L \left(u^{0\prime}(x)\right)^2 dx \right) \right] \exp\left[\frac{1}{2} \beta L \Phi(0) \left(\alpha + \left(\frac{\pi}{L}\right)^2 \right)^{-1} T \right].$$

□

Let us formulate the main result.

Theorem. *The inequality*

$$\left(\|\left(\Delta \boldsymbol{u}_n(t)\right)'\|_n^2 + \|\Delta \boldsymbol{u}_n(t)\|_{Q_n^4}^2 + \alpha \|\Delta \boldsymbol{u}_n(t)\|_{Q_n^2}^2\right)^{\frac{1}{2}} \le \frac{c(t)}{n^{p-1}}, \tag{47}$$

where $c(t)$ *is defined below, is fulfilled for the error of the Galerkin method.*

Proof. By a scalar multiplication of (25) by $2\left(\Delta \boldsymbol{u}_n(t)\right)'$ we obtain

$$F_n'(t) = \frac{1}{2}\beta L\left[\|\Delta \boldsymbol{u}_n(t)\|_{Q_n^2}^2 \left(\|\boldsymbol{u}_n(t)\|_{Q_n^2}^2\right)' + 2\left(\|p_n \boldsymbol{u}(t)\|_{Q_n^2}^2 - \right.\right.$$
$$\left.\left. - \|\boldsymbol{u}_n(t)\|_{Q_n^2}^2\right)\left(Q_n^2 p_n \boldsymbol{u}(t), \left(\Delta \boldsymbol{u}_n(t)\right)'\right)_n\right] + 2\left(\boldsymbol{z}_n(t), \left(\Delta \boldsymbol{u}_n(t)\right)'\right)_n, \tag{48}$$

where

$$F_n(t) = \|(\Delta \boldsymbol{u}_n(t))'\|_n^2 + \|\Delta \boldsymbol{u}_n(t)\|_{Q_n^4}^2 + \left(\alpha + \frac{1}{2}\beta L \|\boldsymbol{u}_n(t)\|_{Q_n^2}^2\right)\|\Delta \boldsymbol{u}_n(t)\|_{Q_n^2}^2. \tag{49}$$

Let us estimate some terms from the right-hand part of relation (48). For this we will have to make use of (17). By (34) and (35) we get

$$\left(\|\boldsymbol{u}_n(t)\|_{Q_n^2}^2\right)' \le \|\boldsymbol{u}_n'(t)\|_n^2 + \|\boldsymbol{u}_n(t)\|_{Q_n^4}^2 \le \Phi_n(t) = \Phi_n(0). \tag{50}$$

Further by virtue of (20), (21), (27) and (33) we see that

$$\left|\|p_n \boldsymbol{u}(t)\|_{Q_n^2}^2 - \|\boldsymbol{u}_n(t)\|_{Q_n^2}^2\right| = \left(\frac{\pi}{L}\right)^2 \sum_{i=1}^n i^2 \, |u_i^2(t) - u_{ni}^2(t)| \le$$
$$\le \sqrt{2}\left(\|p_n \boldsymbol{u}(t)\|_{Q_n^2} + \|\boldsymbol{u}_n(t)\|_{Q_n^2}\right)\|\Delta \boldsymbol{u}_n(t)\|_{Q_n^2} \le \sqrt{2}(c_0 + c_2)\|\Delta \boldsymbol{u}_n(t)\|_{Q_n^2}. \tag{51}$$

Finally, again using (27) we find

$$\left|\left(Q_n^2 p_n \boldsymbol{u}(t), \left(\Delta \boldsymbol{u}_n(t)\right)'\right)_n\right| \le \|p_n \boldsymbol{u}(t)\|_{Q_n^4} \|\left(\Delta \boldsymbol{u}_n(t)\right)'\|_n \le c_1 \|\left(\Delta \boldsymbol{u}_n(t)\right)'\|_n. \tag{52}$$

Relations (48) – (52) together with (17), (26) and (39) allow us to conclude that

$$F_n(t) = \int_0^t F_n'(\tau)d\tau \le \frac{c_3^2 T}{n^{2(p-1)}} + \max(c_4, c_5)\int_0^t F_n(\tau)d\tau,$$

where

$$c_4 = 1 + \nu, \quad c_5 = \left(\alpha + \left(\frac{\pi}{L}\right)^2\right)^{-1}\left(\nu + \frac{1}{2}\beta L \Phi_n(0)\right), \quad \nu = \frac{L}{\sqrt{2}} c_1 \beta (c_0 + c_2). \tag{53}$$

Applying the Gronwall inequality and definition (49), we obtain the proven inequality (47) together with the formula for the coefficient $c(t)$

$$c(t) = c_3 \sqrt{T e^{\max(c_4, c_5)t}}.$$

□

To conclude, note that if we weaken the accuracy requirement, relations (53) can be simplified. By virtue of (37) and (38), $\Phi_n(0)$ and c_2 in (53) can be replaced by $\Phi(0)$ and c_0.

მათემატიკა

მიახლოებითი ალგორითმი ძელის ერთი არაწრფივი განტოლებისათვის

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ნაშრომში დასმულია საწყის-სასაზღვრო ამოცანა კირხჰოფის ინტეგრო-დიფერენციალური განტოლებისათვის, რომელიც აღწერს ძელის დინამიკურ მდგომარეობას. სივრცული და დროის ცვლადების მიმართ ამონახსნის მიახლოებისათვის გამოყენებულია გალიორკინის მეთოდი და სხვაობიანი სქემა. დისკრეტიზების შედეგად მიღებული არაწრფივ განტოლებათა სისტემა ამოიხსნება იაკობის იტერაციული მეთოდის საშუალებით. შეფასებულია გალიორკინის მეთოდის ცდომილება.

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