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Mathematics

On Unconditional Convergence of Series in Banach Spaces with Unconditional Basis

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ABSTRACT. Characterization of the Banach spaces isomorphic to the Banach space c_0 is obtained in terms of unconditionally converging series.

Key words: *Banach lattices, unconditional basis, unconditionally converging series, Sylvester series.*

Let X be a real Banach space and X^* be its topological dual. We remind that a series $\sum_k a_k$ in X converges unconditionally if each of its rearrangements $\sum_k a_{\pi(k)}$ converges in the norm of X. It is not difficult to see that the series $\sum_k a_k$ converges unconditionally in X if and only if $\sum_k \left|\left\langle x^*, a_k \right\rangle\right| < \infty$ for all $x^* \in X^*$ and $\lim_n \sup_{\|x^*\| \le 1} \sum_{k \ge n} \left|\left\langle x^*, a_k \right\rangle\right| = 0$.

Let, in addition, the elements of the Banach space X be partially ordered and the order be compatible with the norm of the Banach space, i.e. the Banach space X is a Banach lattice. Typical examples of Banach lattices are the functional Banach spaces $C(0,1), L_p(0,1)$. As is known, in Banach lattices there exists the notion of the modulus of elements defined as follows: for $x \in X$ the modulus $|x|$ of x is the lowest upper bound of x and $-x$. It is easily seen that from the convergence of the series $\sum_k |a_k|$ in X there follows the unconditional convergence of the series $\sum_k a_k$. The inverse statement is trivial in the case of finite-dimensional Banach lattices because of the equivalence of unconditional and absolute convergence of series in this case. For the infinite-dimensional case this statement was apparently first investigated by W. Sierpinski. In particular, he proved in 1910 that a series $\sum_k f_k$ of bounded real functions defined on non-empty set T is unconditionally uniformly convergent (i.e. it is uniformly convergent regardless of the ordering of its terms) if and only if the series $\sum_k |f_k|$ is uniformly convergent ([1], see also [2] and [3], p. 89). The family $\mathrm{B}(T)$ of all bounded real functions on a set T with the natural ordering (i.e. $x \le y$ whenever $x(t) \le y(t)$ for each $t \in T$) and with the natural norm $\|x\| = \sup_{t \in T} |x(t)|$ is a Banach lattice. The result of Sierpinski can be formulated as follows: the series $\sum_k x_k$ in $\mathrm{B}(T)$ is unconditionally convergent if and only if the series $\sum_k |x_k|$ is convergent. It is interesting to characterize the class of Banach lattices for which the inverse statement, proved by Sierpinski for the Banach lattice $\mathrm{B}(T)$, is true.

In the present paper we investigate this problem for the class of Banach spaces with unconditional basis which is a particular case of general Banach lattices.

Let a Banach space X have an unconditional basis (φ_i) and (φ_i^*) be the corresponding biorthogonal sequence of linear bounded functionals. Any unconditional basis induces in a natural way a partial order in X: $\sum_i \langle \varphi_i^*, x\rangle \varphi_i \geq 0 \Leftrightarrow \langle \varphi_i^*, x\rangle \geq 0$ for all i, $x \in X$, and the modulus of the element $x = \sum_i \langle \varphi_i^*, x\rangle \varphi_i$ is $|x| = \sum_i \left|\langle \varphi_i^*, x\rangle\right| \varphi_i$. It is easy to see that for the series $\sum_k a_k$ in X the condition

$$\sum_i \left(\sum_k \left|\langle \varphi_i^*, a_k\rangle\right| \right) \varphi_i \quad \text{converges in } X \tag{1}$$

is equivalent to convergence of the series $\sum_k |a_k|$.

Theorem. *Let X be an infinite-dimensional Banach space with unconditional basis (φ_i) and (φ_i^*) be the corresponding biorthogonal sequence of linear bounded functionals. Then the following statements are equivalent.*

(i) *Unconditional convergence of a series $\sum_k a_k$ in X implies convergence of the series $\sum_k |a_k|$ in X.*

(ii) *X, as a Banach lattice, is order isomorphic to c_0.*

(iii) *X, as a Banach space, is isomorphic to c_0.*

Before the proof of this theorem we remind that c_0 denotes the Banach space of real numerical sequences converging to zero, and the order in c_0 is induced by the natural basis (the sequence of unit vectors). The Banach lattices X and Y are order isomorphic if X and Y are isomorphic as Banach spaces and isomorphism between them can be chosen by positive operator with a positive inverse (operator $T : X \to Y$ is positive if $Tx \geq 0$ for all $x \geq 0, x \in X$). Clearly, isomorphic Banach lattices are isomorphic as Banach spaces (the converse, clearly, is not always valid). More information on Banach lattices can be found, for example, in [4].

The proof of the theorem is based on the use of series constructed by Sylvester matrices (we call them Sylvester series; such series were considered in [5-8]).

The Sylvester matrix $S^{(n)} = \left[s_{ki}^{(n)}\right]$ is defined by the following recurrence relations:

$$S^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad S^{(n)} = \begin{bmatrix} S^{(n-1)} & S^{(n-1)} \\ S^{(n-1)} & -S^{(n-1)} \end{bmatrix}, \quad n = 2,3,\ldots.$$

Note the following obvious property of the Sylvester matrices: for any sequence of real numbers $\beta_1, \beta_2, \ldots, \beta_n$ the following equality is true:

$$\sum_{k=1}^{2^n} \left(\sum_{i=1}^{2^n} s_{ki}^{(n)} \beta_i \right)^2 = 2^n \sum_{i=1}^{2^n} \beta_i^2 . \tag{2}$$

Let X be a Banach space (not necessarily with basis), (a_i) be a sequence in X and (δ_n) be a sequence of real numbers. Denote by $I_n = \{2^n - 1, 2^n, \ldots, 2^{n+1} - 2\}$, $n = 1,2,\ldots$, the partition of positive integers. Furthermore, for any positive integer k we denote by $\bar{k}$ the positive integer uniquely defined by the relations:

$$\bar{k} = k - 2^n + 2, \quad k \in I_n, \quad n = 1,2,\ldots. \tag{3}$$

For any positive integer n and for every Sylvester matrix $S^{(n)} = \left[s_{ki}^{(n)}\right]$ we construct the sequence of elements (d_k) in X by the equalities

$$d_k = \delta_n \sum_{i \in I_n} s_{\bar{k}\bar{i}}^{(n)} a_{\bar{i}}, \qquad k \in I_n, \quad n = 1,2,\ldots.$$

The series composed by d_k we will call the Sylvester series.

The following lemma gives the sufficient conditions for the unconditional convergence of the Sylvester series.

Lemma. *If*

$$(a)\ \sum_n \delta_n 2^n \left(\sum_{i\in I_n} \langle x^*, a_i\rangle^2\right)^{1/2} < \infty \quad \textit{for every} \quad x^* \in X^*$$

and

$$(b)\ \lim_{m\to\infty} \sup_{\|x^*\|\le 1} \sum_{n\ge m} \delta_n 2^n \left(\sum_{i\in I_n} \langle x^*, a_i\rangle^2\right)^{1/2} = 0,$$

then the Sylvester series $\sum_k d_k$ *converges unconditionally in* X.

Proof. Using the closed graph theorem, it is easy to check that (a) implies the inequality $\sup_{\|x^*\|\le 1} \sum_n \delta_n 2^n \left(\sum_{i\in I_n} \langle x^*, a_i\rangle\right)^{1/2} < \infty$. Taking into account the equality $\sum_k = \sum_n \sum_{k\in I_n}$ and using the Cauchy inequality in the internal sum and then the relation (2), we get:

$$\sum_{k\ge l} \left|\langle x^*, d_k\rangle\right| \le \sum_{n\ge n_l} \delta_n 2^n \left(\sum_{i\in I_n} \langle x^*, a_i\rangle^2\right)^{1/2},$$

where n_l is determined uniquely by the conditions $2^{n_l} - 1 \le l \le 2^{n_l+1} - 2,\ l = 1,2,\ldots$. Hence we have the unconditional convergence of the series $\sum_k d_k$.

Corollary. *The Sylvester series* $\sum_k d_k$ *converges unconditionally in* X *if one of the following conditions is fulfilled:*

$$(i)\ \sum_n \delta_n 2^n \sup_{\|x^*\|\le 1} \left(\sum_{i\in I_n} \langle x^*, a_i\rangle^2\right)^{1/2} < \infty,$$

or

$$(ii)\ \sum_n \delta_n 2^n \max_{i\in I_n} \|a_i\|^{1/2} \max_{\vartheta_i=\pm 1} \left\|\sum_{i\in I_n} \vartheta_i a_i\right\|^{1/2} < \infty. \tag{4}$$

Proof. It is clear that condition (i) implies fulfillment of conditions (a) and (b) of Lemma. The proof of the sufficiency of condition (ii) for unconditional convergence of the series $\sum_k d_k$ follows from the following elementary inequality for the case $p = 1$: let $x_1, x_2, \ldots x_m,\ m \ge 1$, be elements in the unit ball of a normed space X and x^* be an element in the unit ball of the dual space X^*, then for any $p \ge 1$ the following inequality holds

$$\sum_{i=1}^m \left|\langle x^*, x_i\rangle\right|^p \le \max_{\vartheta_i=\pm 1} \left\|\sum_{i=1}^m \vartheta_i x_i\right\|.$$

The proof of the inequality can be given from the following chain of relations:

$$\sum_{i=1}^m \left|\langle x^*, x_i\rangle\right|^p \le \sum_{i=1}^m \left|\langle x^*, x_i\rangle\right| = \sum_{i=1}^m \langle x^*, x_i\rangle \operatorname{sgn}\langle x^*, x_i\rangle \le \max_{\vartheta_i=\pm 1} \left\|\sum_{i=1}^m \vartheta_i x_i\right\|.$$

Now we can prove the main result of the paper.

Proof of the Theorem. $(i) \Rightarrow (ii)$. At first we note that if the spaces X and c_0 are isomorphic, then they are order isomorphic as well. Indeed, let $T : X \to c_0$ be an isomorphism operator. Since in c_0 all normed unconditional bases are equivalent (see [9], p. 71), there exists an isomorphism operator $U : c_0 \to c_0$ such that $U(T\varphi_i / \|T\varphi_i\|) = e_i$ for any indices i, where (e_i) is the natural basis in c_0. Then the composition UT clearly is an operator which realizes the order isomorphism between X and c_0. To prove (ii) suppose, contrary to our claim, that X is not isomorphic to c_0. Without loss of generality we can suppose that (φ_i) is a normed basis. By our presumption, the functional $\lambda(n) = \left\|\sum_{i=1}^n \varphi_i\right\|$ will be non-bounded with respect to n. Consequently there exists a strictly increasing sequence of positive numbers (n_l) such that $\left\|\varphi_1 + \varphi_2 + \ldots + \varphi_{2^{n_l}}\right\| \ge l^4$ for all $l,\ l = 1,2,\ldots$. We construct a Sylvester series in the following way: for all $n \ne n_l$ suppose $d_k = 0,\ k \in I_n$, and for $n = n_l$

$$d_k = \delta_l \sum_{i=1}^{2^{n_l}} s_{\bar{k}\,i}^{(n_l)}\, \varphi_i, \quad k \in I_{n_l}, \quad l = 1,2,\ldots,$$

where, as above, the dash on the index k is defined by (3), and the sequence (δ_l) fulfills the conditions

$$\sum_{l=1}^{\infty} \delta_l\, 2^{n_l} \left\| \sum_{i=1}^{2^{n_l}} \varphi_i \right\|^{1/2} < \infty \quad \text{and} \quad \lim_{l\to\infty} \delta_l\, 2^{n_l} \left\| \sum_{i=1}^{2^{n_l}} \varphi_i \right\| \neq 0 . \tag{5}$$

We can choose the sequence (δ_l), for example, as follows:

$$\delta_l = 2^{-n_l} \left\| \sum_{i=1}^{2^{n_l}} \varphi_i \right\|^{-1/2} l^{-2}, \quad l = 1,2,\ldots .$$

With this choice the series $\sum_k d_k$ converges unconditionally, since the condition (4) is fulfilled by the first condition in (5). On the other hand, the series $\sum_k |d_k|$ does not converge. Indeed, if the series $\sum_k |d_k|$ does converge, then we have $\sum_i \left(\sum_k \left| \langle \varphi_i^*, d_k \rangle \right| \right) \varphi_i = \sum_l \delta_l\, 2^{n_l} \left(\sum_{i=1}^{2^{n_l}} \varphi_i \right)$ and the last series cannot be converging by the second condition in (5). The proof of this implication is finished.

The implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are clear, and the proof of the theorem is finished.

Going back to condition (1), it is not difficult to see that in the space l_1, this condition implies an absolute convergence of the series $\sum_k a_k$. Let now X be a Banach space with an unconditional basis (φ_i) and (φ_i^*) be the corresponding biorthogonal sequence of linear bounded functionals. Suppose that X has the following property: if the series $\sum_k a_k$ satisfies condition (1) then it converges absolutely (i.e. $\sum_k \|a_k\| < \infty$). If this is the case, then taking $\beta_k\, \varphi_k$ as a_k, where β_k are real numbers, (φ_i) will be an absolute basis and therefore X will be isomorphic to the space l_1. Since in l_1 all normed unconditional bases are equivalent (see [9], p. 71), we get the order isomorphism. Therefore, the following assertion is valid.

Statement. Let X be an infinite-dimensional Banach space with unconditional basis (φ_i) and (φ_i^*) be the corresponding biorthogonal sequence of linear bounded functionals. Then the following assertions are equivalent.

(i) From the convergence of the series $\sum_k |a_k|$ in X there follows the absolute convergence of the series $\sum_k a_k$.

(ii) X, as a Banach lattice, is order isomorphic to l_1.

(iii) X, as a Banach space, is isomorphic to l_1.

In the paper [10] of P. Kostyrko it is proved that in L-spaces the assertion (i) of this Statement is valid. In this connection it would be interesting to characterize those Banach lattices in which the assertion (i) of this Statement is valid.

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