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# Complete Semigroups of Binary Relations whose Set of All Idempotents is a Semigroup

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**ABSTRACT. In this paper we investigate a complete semigroup of binary relations whose set of all idempotents is a semigroup.** 

***Key words:*** *semigroup, binary relations, idempotent elements.*

Let $X$ be an arbitrary nonempty set, $D$ be a complete $X$-semilattice of unions, i.e. $D$ be some nonempty set of subsets from $X$ which is closed with respect to the set-theoretic union of elements from $D$, and $f$ be an arbitrary mapping of the set $X$ in the set $D$. To each mapping $f$ we assign a binary relation $\alpha_f$ on the set $X$ which satisfies the conditions

$$\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x)).$$

The set of all such $\alpha_f$ $(f : X \to D)$ is denoted by $B_X(D)$. As is well known, $B_X(D)$ is a subsemigroup of the semigroup $B_X$ called a complete semigroup of binary relations defined by an $X$-semilattice of unions $D$.

Let $\alpha \in B_X(D)$, $Y \subseteq X$ and $Y\alpha = \{x \in X \mid (y, x) \in \alpha$ for some $y \in Y\}$; $V(D,\alpha) = \{Y\alpha \mid Y \in D\}$; $\varnothing \neq D' \subseteq D$ and $N(D, D') = \{Z \in D \mid Z \subseteq Z'$ for any $Z' \in D'\}$. If $N(D, D') \neq \varnothing$, then $\bigcup N(D, D') \in D$ is an exact lower bound of a set $D'$ in $D$. We denote this element by $\Lambda(D, D')$. Note that if the element $\Lambda(D, D')$ exists in the semilattice $D$, then we write $\Lambda(D, D') \in D$.

**Definition 1.** Let $D$ be an arbitrary complete $X$-semilattice of unions, $\alpha \in B_X$ and $Y_T^\alpha = \{x \in X \mid x\alpha = T\}$. If

$$V[\alpha] = \begin{cases} V(X^*,\alpha), & \text{if } \varnothing \notin D, \\ V(X^*,\alpha), & \text{if } \varnothing \in V(X^*,\alpha), \\ V(X^*,\alpha)\cup\{\varnothing\}, & \text{if } \varnothing \notin V(X^*,\alpha) \text{ and } \varnothing \in D, \end{cases}$$

then it is obvious that any binary relation $\alpha$ of the semigroup $B_X$ can always be represented as

$$\alpha = \bigcup_{T\in V[\alpha]}(Y_T^\alpha \times T).$$

In the sequel such a representation of a binary relation $\alpha$ will be called quasinormal.

Note that for a quasinormal representation of a binary relation $\alpha$, not all sets $Y_T^\alpha$ may be different from an empty set. But in this representation the following conditions are always fulfilled:

a) $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$ for any $T, T' \in D$ and $T \neq T'$;

b) $X = \bigcup_{T \in V[\alpha]} Y_T^\alpha$.

**Definition 2.** Let $t \in \breve{D} = \bigcup_{Z \in D} Z$ and $D_t = \{Z \in D \mid t \in Z\}$. We say that a complete $X$-semilattice of unions $D$ is an $XI$-semilattice if it satisfies the following two conditions: $\Lambda(D, D_t) \in D$ for any $t \in \breve{D}$ and $Z = \bigcup_{t \in Z} \Lambda(D, D_t)$ for any nonempty element $Z$ of the semilattice $D$.

**Definition 3.** We say that a complete $X$-semilattice of unions $D$ is nodal if $D = \{Z_1, Z_2, \breve{D}\}$ for some pairwise different subsets $Z_1$, $Z_2$ and $\breve{D}$ of the set $X$ which satisfy the conditions $Z_1 \cap Z_2 = \varnothing$ and $Z_1 \cup Z_2 = \breve{D}$.

The following statements can be found in [1-3].

**Theorem 1.** *Let $\varnothing \notin D$ and $|D| \geq 1$. Then the right zeros of the semigroup $B_X(D)$ are elements from $K_X(D) = \{X \times Z \mid Z \in D\}$* ([1], *p.* 377).

**Theorem 2.** *Let D be an elementary X-semilattice of unions. Then the set of all idempotents of the semigroup $B_X(D)$ forms an idempotent* ([2], Theorem 4, p. 3175).

**Theorem 3.** *Let D and $I_X(D)$ be respectively a nodal X-semilattice of unions and a subsemigroup of the semigroup $B_X(D)$ generated by all idempotent elements of the semigroup $B_X(D)$. Then $I_X(D)$ is a semigroup of idempotent elements* ([2], Theorem 2, p. 3208).

**Theorem 4.** *A binary relation $\varepsilon \in B_X(D)$ is a right unit of this semigroup if and only if ε is idempotent and $D = V(D, \varepsilon)$* ([3], Theorem 2.1, p. 4281).

**Theorem 5.** *Let D be a complete X-semilattice of unions, $t \in \breve{D} = \cup D$ and $D_t = \{Z \in D \mid t \in Z\}$. The semigroup $B_X(D)$ has right units if and only if*

1) $\Lambda(D, D_t \in D)$ *for all* $t \in \breve{D}$ *and*

2) $Z' = \bigcup_{t \in Z'} \Lambda(D, D_t)$ *for any nonempty* $Z'$ *in D* ([3], Theorem 2.6, p. 4286).

**Lemma 1.** *Assume that the X-semilattice of unions D does not contain a three-element chain. Then the semilattice D satisfies at least one of the following conditions:*

a) $D = \{Z_1\}$ *for some subset* $Z_1$ *of the set X;*

b) $D = \{Z_1, Z_2\}$ *for some subsets* $Z_1$ *and* $Z_2$ *of the set X which satisfy the condition* $Z_1 \subset Z_2$;

c) $D = \{Z_1, Z_2, \breve{D}\}$ *is a nodal X-semilattice of unions;*

d) *D is an elementary X-semilattice of unions.*

**Proof.** Assume that the $X$-semilattice of unions $D$ does not contain a three-element chain.

a′) The statements a) and b) are valid if $|D| = 1$ or $|D| = 2$.

b′) Let $|D| \geq 3$ and $\breve{D}$ be the largest element of the semilattice $D$. If $Z$ and $Z'$ are various elements of the semilattice $D$ that are different from the element $\breve{D}$, then, by assumption, we have $Z, Z' \in \breve{D}$ and $Z \subseteq Z \cup Z' \subseteq \breve{D}$.

1) It is obvious that if the conditions $Z \neq Z \cup Z'$ and $Z \cup Z' \neq D'$ are fulfilled, then the inclusion $Z \subseteq Z \cup Z' \subseteq \breve{D}$ implies the inclusion $Z \subset Z \cup Z' \subset \breve{D}$. However, the latter inclusion contradicts the assumption that the semilattice $D$

does not contain a three-element chain. Therefore $Z = Z \bigcup Z'$ or $Z \bigcup Z' = \breve{D}$.

If $Z = Z \bigcup Z'$, then $Z' \subset Z$, since, by assumption, $Z \neq Z'$. Now, taking into account the inclusion $Z \subset \breve{D}$, we obtain $Z' \subset Z \subset \breve{D}$. However the latter inclusion also contradicts the assumption that the semilattice $D$ does not contain a three-element chain. The obtained contradiction shows that $Z \bigcup Z' = \breve{D}$.

We have established that $Z \bigcup Z' = \breve{D}$ for any elements $Z, Z' \in D$.

2) Now, if $Z_1 \bigcap Z_2 = \varnothing$ for some elements $Z_1, Z_2 \in D$, then for any element $Z \in D \setminus \{Z_1, Z_2\}$ we have $Z_1 \bigcup Z = Z_1 \bigcup Z_2 = \breve{D}$ and $Z_2 \bigcup Z = Z_1 \bigcup Z_2 = \breve{D}$. From this we respectively obtain $Z_2 \subseteq Z$, $Z_1 \subseteq Z$ and $Z_1 \bigcup Z_2 = \breve{D} \subseteq Z$. Therefore $Z = \breve{D}$, since $\breve{D}$ is the largest element of the semilattice $D$.

Thus $D = \{Z_1, Z_2, \breve{D}\}$ is a nodal $X$-semilattice of unions.

The statement c) of the lemma is proved.

3) If the set-theoretic intersection of any two elements of the semilattice $D$ is nonempty, then statement 1) immediately implies that $D$ is an elementary $X$-semilattice of unions.

The statement d) of the lemma is proved.

The proof of the lemma is completed.

Note that according to Lemma 1 the diagrams of $X$-semilattices of unions which do not contain a three-element chain have one of the forms 1, 2, 3 or 4 shown in the Figure.

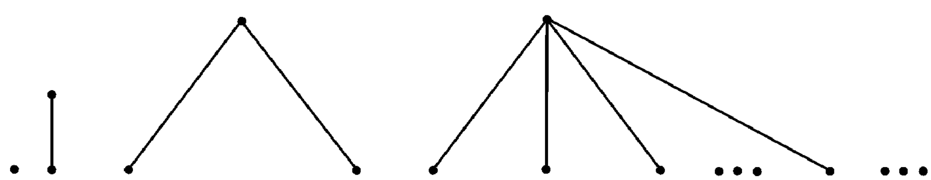

Figure.

To find all idempotents of the semigroup $B_X(D)B$, we should first define by virtue of Theorem 5 all $XI$-subsemilattices of the semilattice $D$. After that we find the right units of all complete semigroups of binary relations which are defined by these $XI$-semilattices of unions. Combining the sets of all right units, we obtain the set of all idempotents of the considered semigroup.

It can be easily shown that only semilattices having a representation of one of the forms $D' = \{T\}$, $D'' = \{T', T''\}$ and $D''' = \{Z, Z', Z \bigcup Z'\}$, where $T, T', T'', Z, Z', Z'' \in D$, $T' \subset T''$, $Z \setminus Z' \neq \varnothing$, $Z' \setminus Z \neq \varnothing$ and $Z \bigcap Z' = \varnothing$ (see [4]) can be $XI$-subsemilattices of the $X$-semilattice of unions $D$.

The diagrams of the considered $XI$-semilattices of unions have one of the forms 1, 2 and 3 shown in the Fig. Quasinormal representations of the elements of the semigroups $B_X(D')$, $B_X(D'')$ and $B_X(D''')$ have respectively the form:

a) $\alpha = X \times T$;

b) $\beta = \left(Y_{T'}^{\beta} \times T'\right) \bigcup \left(Y_{T''}^{\beta} \times T''\right)$ for some nonintersecting subsets $Y_{T'}^{\beta}$, $Y_{T''}^{\beta}$ of the set $X$ which satisfy the conditions $Y_{T'}^{\beta}, Y_{T''}^{\beta} \notin \{\varnothing\}$ and $Y_{T'}^{\beta} \bigcup Y_{T''}^{\beta} = X$;

c) $\delta = \left(Y_{Z}^{\delta} \times Z\right) \bigcup \left(Y_{Z'}^{\delta} \times Z'\right) \bigcup \left(Y_{Z''}^{\delta} \times Z''\right)$ for some pairwise nonintersecting subsets $Y_{Z}^{\delta}$, $Y_{Z'}^{\delta}$ and $Y_{Z''}^{\delta}$ of the set $X$ which satisfy the conditions $Y_{Z}^{\delta}, Y_{Z'}^{\delta} \notin \{\varnothing\}$ and $Y_{Z}^{\delta} \bigcup Y_{Z'}^{\delta} \bigcup Y_{Z''}^{\delta} = X$.

1) Note that $\alpha \circ \alpha = \alpha$;

2) $\beta \circ \beta = \beta$ if and only if $Y_{T'}^{\beta} \supseteq T'$ and $Y_{T''}^{\beta} \bigcap T'' \neq \varnothing$;

3) $\delta \circ \delta = \delta$ if and only if $Y_{Z}^{\delta} \supseteq Z$ and $Y_{Z'}^{\delta} \supseteq Z'$ ([2], Theorem 1, p. 3206).

**Lemma 2.** *If the semilattice $D$ satisfies the statement* a) *of Lemma* 1, then the semigroup $B_X(D)$ is a unit semigroup.

**Proof.** If the semilattice $D$ satisfies the statement a) of Lemma 1, then $B_X(D)=\{X\times Z_1\}$, where $Z_1$ is some subset of the set $X$ and $(X\times Z_1)\circ(X\times Z_1)=X\times Z_1$.

The lemma is proved.

**Lemma 3.** *If the semilattice D satisfies the statement* b) *of Lemma* 1, *then the set of all idempotent elements of the semigroup* $B_X(D)$ *forms a subsemigroup of the semigroup* $B_X(D)$.

**Proof.** Let $\alpha$ be an arbitrary element of the semigroup $B_X(D)$. Then a quasinormal representation of a binary relation $\alpha$ has the form

$$\alpha=\left(Y_{Z_1}^{\alpha}\times Z_1\right)\cup\left(Y_{Z_2}^{\alpha}\times Z_2\right),$$

where

$$Y_{Z_1}^{\alpha},Y_{Z_2}^{\alpha}\subseteq X\ ,\quad Y_{Z_1}^{\alpha}\cap Y_{Z_2}^{\alpha}=\varnothing \text{ and } Y_{Z_1}^{\alpha}\cup Y_{Z_2}^{\alpha}=X\ .$$

A relation $\alpha$ is idempotent if and only if it satisfies at least one of the following two conditions:

1) $\alpha=X\times Z_1$ or $\alpha'=X\times Z_2$;

2) $\beta=\left(Y_{Z_1}^{\beta}\times Z_1\right)\cup\left(Y_{Z_2}^{\beta}\times Z_2\right)$ for some subsets $Y_{Z_1}^{\beta}$ and $Y_{Z_2}^{\beta}$ of the set $X$ which satisfy the conditions $Y_{Z_1}^{\beta},Y_{Z_2}^{\beta}\notin\{\varnothing\}$, $Y_{Z_1}^{\beta}\supseteq Z_1$ and $Y_{Z_2}^{\beta}\cap Z_2\neq\varnothing$.

Now let $\beta'=\left(Y_{Z_1}^{\beta'}\times Z_1\right)\cup\left(Y_{Z_2}^{\beta'}\times Z_2\right)$ for some subsets $Y_{Z_1}^{\delta}$ and $Y_{Z_2}^{\delta}$ of the set $X$ which satisfy the conditions $Y_{Z_1}^{\beta'},Y_{Z_2}^{\beta'}\notin\{\varnothing\}$, $Y_{Z_1}^{\beta'}\supseteq Z_1$ and $Y_{Z_2}^{\beta'}\cap Z_2\neq\varnothing$.

We will prove that the set of all idempotent elements of the semigroup $B_X(D)$ forms a subsemigroup of the semigroup $B_X(D)$. For this we have to prove that the products $\alpha\circ\alpha'$, $\alpha\circ\beta$, $\beta\circ\alpha$ and $\beta\circ\beta'$ are idempotent elements of the semigroup $B_X(D)$.

To begin with, we observe that $\beta\circ\beta=\beta$, $\beta'\circ\beta'=\beta'$ and $V(D,\beta)=V(D,\beta')=D$. From this we conclude by virtue of Theorem 4 that the binary relations $\beta$ and $\beta'$ are right units of the semigroup $B_X(D)$. Hence it follows that $\alpha\circ\beta=\alpha$ and $\beta\circ\beta'=\beta$. Moreover, by virtue of Theorem 1 the binary relations $\alpha$ and $\alpha'$ are right zeros of the semigroup $B_X(D)$. Therefore $\alpha\circ\alpha'=\alpha'$ and $\beta\circ\alpha=\alpha$.

Thus $\alpha\circ\alpha'=\alpha'$, $\alpha\circ\beta=\alpha$, $\beta\circ\alpha=\alpha$ and $\beta\circ\beta'=\beta$ are idempotent elements of the semigroup $B_X(D)$, since, by assumption, $\alpha=\alpha\circ\alpha$, $\beta\circ\beta=\beta$ and $\alpha'\circ\alpha'=\alpha'$.

The lemma is proved.

**Theorem 6.** *The set of all idempotents of the semigroup* $B_X(D)$ *forms a subsemigroup of this semigroup if and only if the complete X-semilattice of unions D does not contain a subsemilattice which is a three-element chain.*

**Proof.** Note that the set of all idempotents of the semigroup $B_X(D)$ forms a subsemigroup of this semigroup and $D'=\{Z_1,Z_2,Z_3\}$ is a subsemilattice of the semilattice $D$ such that $Z_1\subset Z_2\subset Z_3$. From the definition of complete semigroups of binary relations it immediately follows that $B_X(D')\subseteq B_X(D)$. Now let

$$\alpha=(Z_1\times Z_1)\cup((X\setminus Z_1)\times Z_3),\quad \beta=(Z_2\times Z_2)\cup((X\setminus Z_2)\times Z_3)\ .$$

Then $\alpha,\beta\in B_X(D')$ and the equalities $\alpha\circ\alpha=\alpha$, $\beta\circ\beta=\beta$ are valid, since

$$Z_1\supseteq Z_1,\ (X\setminus Z_1)\cap Z_3\neq\varnothing \text{ and } Z_2\supseteq Z_2,\ (X\setminus Z_2)\cap Z_3\neq\varnothing\ .$$

Also,

$$\alpha\circ\beta=(Z_1\times Z_2)\cup((X\setminus Z_1)\times Z_3) \text{ and } (\alpha\circ\beta)\circ(\alpha\circ\beta)\neq\alpha\circ\beta\ .$$

However the latter inequality contradicts the assumption that the set of all idempotent elements of the semigroup $B_X(D)$ forms a subsemigroup of this semigroup. The obtained contradiction shows that the $X$-semilattice of unions $D$ cannot contain a three-element chain.

On the other hand, if the $X$-semilattice of unions $D$ does not contain a three-element chain, then, according to Lemma 1, the diagrams of $X$-semilattices of unions which do not contain a three-element chain have one of forms 1, 2, 3 and 4 shown in Fig. 1. From Lemmas 2, 3 and Theorems 2, 3 it immediately follows that the set of all idempotents of the semigroup $B_X(D)$, which are defined by semilattices whose diagrams coincide at least with one of those shown in the Figure, forms a subsemigroup of the semigroup $B_X(D)$.

The theorem is proved.

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# ბინარულ მიმართებათა სრული ნახევარჯგუფები, რომელთა ყველა იდემპოტენტების სიმრავლე ქვენახევარჯგუფებია

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