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Physics

On the Quantum Relativistic Origin of the Coulomb Potential

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ABSTRACT. It is proved that the Witten's superalgebra based on Dirac's *K*-operator and anticommuting with it Johnson-Lippmann like operator may serve as a theoretical ground of the origin of Coulomb potential. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *hidden symmetry, Dirac equation, Coulomb potential.*

The Coulomb potential plays a fundamental role in nature. It determines the principal electrostatic interaction between electrically charged point particles. Gravitational potential coincides by its form with the Coulomb one: both of them give force, which is decreased quadratically with distance. Forces of this kind are responsible for periodic motion of planets, on the one hand, and for atomic level structure, on the other.

Both potentials are obtained on experimental grounds: by determination of static force between charged particles (Coulomb experiment) and by observation of planetary motion (Kepler's laws). As is known there is no other theoretical justification of this potential (except a one-photon exchange mechanism in QFT, which rests on experiments as well).

In the works [1-3] we have studied the so-called hidden symmetry of Coulomb interaction, which determines an accidental degeneracy of levels of hydrogen problem. Relativistic quantum mechanics (Dirac equation) clears up several remarkable facts, which give a possibility to look at the Kepler problem from a fresh point of view and establish the origin of Coulomb potential theoretically, to our opinion.

For justification of our concepts let us consider the following Dirac Hamiltonian

$$H = \vec{\alpha} \cdot \vec{p} + \beta m + V(r), \tag{1}$$

where $V(r)$ is an arbitrary central potential, which in this form is the 4$^{\text{th}}$ component of a Lorentz-vector. Together with the total momentum $\vec{J} = \vec{l} + \frac{1}{2}\vec{\Sigma}$ there is the so-called Dirac's *K*-operator, defined as

$$K = \beta\left(\vec{\Sigma} \cdot \vec{l} + 1\right), \tag{2}$$

which commutes with *H*.

Therefore eigenstates of *H* are also labeled by eigenvalues of *K*-operator, $\kappa = \pm(j + 1/2)$. Let us remark that the positive values of κ correspond to the aligned spin, when $j = l + 1/2$, or to the states $(s_{1/2}, p_{3/2}, etc.)$, while the negative values: correspond to unaligned spin, $j = l - 1/2$, or to the states $(p_{1/2}, d_{3/2}, etc.)$. When there is a degeneracy by the signs of κ, then the Lamb shift $(s_{1/2} - p_{1/2})$ is forbidden. But really this shift is observed experimentally and is explained by QED radiative corrections, or equivalently, by modification of Coulomb potential [4].

It is interesting to investigate the physical meaning of $\kappa \to -\kappa$ degeneracy. For this reason we need to introduce a generator, which interchanges these signs. It is evident that this generator, say A, must be anticommuting with K:

$$\{A, K\} = 0. \tag{3}$$

Let us mention that if such an operator has been introduced, the so-called Witten's superalgebra S(2) [5] appears immediately. For this purpose it is sufficient to identify supercharges as follows:

$$Q_1 = A, \qquad\qquad Q_2 = i\frac{AK}{|\kappa|}. \tag{4}$$

It is clear that

$$\{Q_1, Q_2\} = 0, \tag{5}$$

and moreover

$$Q_1^2 = Q_2^2 \equiv h$$

The last operator plays the role of Witten's Hamiltonian [5].

If we want the Witten's algebra to be a symmetry of the Dirac Hamiltonian, one is to require that the generators of this algebra Q_i $(i = 1,2)$ commute with the Hamiltonian (1), or equivalently

$$[A, H] = 0. \tag{6}$$

In order to construct such an operator, which at the same time anticommutes with K, we can use a theorem, proved by us earlier [2]. According to this theorem any operator of kind $(\vec{\Sigma} \cdot \vec{V})$, where $\vec{V}$ is a vector with respect to $\vec{l}$ and is perpendicular to it, i.e. $(\vec{l} \cdot \vec{V}) = (\vec{V} \cdot \vec{l}) = 0$, is anticommuting with K:

$$\{(\vec{\Sigma} \cdot \vec{V}), K\} = 0. \tag{7}$$

It is clear that the most general class of required operators would be of the form $\hat{O}(\vec{\Sigma} \cdot \vec{V})$, where $[\hat{O}, K] = 0$. As is mentioned above, the Witten's algebra will be a symmetry of Dirac's Hamiltonian, if one constructs such combination of K-odd operators that commute with H. It is shown in our paper [3] that the following combination

$$A = (\vec{\Sigma} \cdot \hat{\vec{r}}) - \frac{i}{ma} K(\vec{\Sigma} \cdot \vec{p}) + \frac{i}{mr} K\gamma^5 \tag{8}$$

commutes with H if and only if the potential is Coulombic: $V(r) = -\frac{a}{r}$. In this case operator (8) reduces to the form of Johnson and Lippmann (JL) [6]

$$A = \gamma^5 \left\{ \vec{\alpha} \cdot \hat{\vec{r}} - \frac{i}{ma} K\gamma^5 (H - \beta m) \right\} \tag{9}$$

It is remarkable to mention that one can include a Lorentz-scalar potential $\beta S(r)$ in a Dirac Hamiltonian or consider even a more general combination:

$$H = \vec{\alpha} \cdot \vec{p} + \beta m + V(r) + \beta S(r). \tag{10}$$

As is well known the non-relativistic (Schroedinger) equation is indifferent with respect to Lorentz-variance properties of potential. Moreover, in non-relativistic quantum mechanics we have an additional symmetry for Coulomb potential. Therefore it is natural to expect that the Hamiltonian (10) has also this additional symmetry and hence the JL operator (9) must be generalized to this case as well.

Indeed, we have shown [3] that the following operator

$$X = x_1\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right) + x_1'\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)H + ix_2K\left(\vec{\Sigma}\cdot\vec{p}\right) + ix_3K\gamma^5 f_1(r) + ix_3'K\gamma^5\beta f_2(r), \tag{11}$$

which is constructed in accordance with our theorem and obeys all the requirements of it, commutes with the Hamiltonian (10) if and only if both potentials $V(r)$ and $S(r)$ are Coulombic. In this case equation (11) simplifies and reduces to the following form [7]:

$$X = \left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)\left(ma_V + Ha_S\right) - iK\gamma^5\left(H - \beta m\right). \tag{12}$$

In the non-relativistic limit this operator reduces to the projection of Laplace-Runge-Lenz (LRL) vector into the spin direction of the considered fermion. After all that, the utilization of Witten's algebra gives us the correct energy spectrum expressions without any reference to the equations of motion.

In conclusion, we see that the symmetry operators, obtained by us, are closely related to the symmetry of LRL vector. Therefore the orbital motion of planets follows the same laws, as the structure of hydrogen atom levels, and all of these results are valid only for Coulomb potential. Moreover, we found a theoretical ground for Coulomb potential: It is a supersymmetry on the microscopic level in the above mentioned meaning, eqs. (4-5). It is also remarkable that the corresponding symmetry operator becomes a generator of definite $(\kappa \to -\kappa)$ transformation only on microscopic level, which means a transformation of spin degrees of freedom. The results underlined above give us a background to prove that the origin of Coulomb potential lies in relativistic quantum mechanics, and a macroscopic manifestation of it is a relic of peculiarities of relativistic quantum features, in other words, a supersymmetry is around us. Let us remark also that these results may be generalized in higher (n>3) spatial dimensions as well [8] if and only if the Coulombic behavior (1/r) is true in any dimensions.

ფიზიკა

კულონური პოტენციალის კვანტურ-რელატივისტური წარმოშობის შესახებ

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