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Mathematics

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General Solutions of Linear Matrix Canonical *m*-Dimensional Equations with Variable Coefficients

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ABSTRACT. This article contains the formulae of general solutions for particular classes of non-homogeneous ordinary linear *m*-dimensional matrix differential equations with variable coefficients. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *matrix differential equations, variable coefficients, general solutions.*

1. Canonical Equation. Problem Statement

Let a linear matrix canonical m-dimensional differential equation be called an equation

$$(p-A_1)\left[(p-A_2)\left[\dots(p-A_{m-1})\left[(p-A_m)X\right]\dots\right]\ \right]=F,\ p=\frac{d}{dt}, \tag{1}$$

where

$$F=F(t),\ A_i=A_i(t),\ i=\overline{1,m},\ t\in I=[t_1,t_2]\subset]-\infty,+\infty[$$

are given $n\times n$ matrices with continuous and continuously differentiable (with respect to t by corresponding number) elements on the interval I.

Definition 1. The solution of equation (1) will be called matrix function $X=X(t)$ defined on the interval I, substitution of which in the equation (1) is admissible as a result of which we get the identity

$$(p-A_1)\left[(p-A_2)\left[\dots(p-A_{m-1})\left[(p-A_m)X\right]\dots\right]\ \right]\equiv F,\ t\in I.$$

Definition 2. Let t_0 be an arbitrary fixed point of the interval I and $X_0,\dots X_{m-1}$ be arbitrary fixed constants of matrix $n\times n$. Matrix function $X(t,C_0,\dots,C_{m-1})$ defined on the interval I and depending on arbitrary constants $C_0,\dots,C_{m-1}$, of matrix $n\times n$ will be called the general solution of equation (1), if $X(t,C_0,\dots,C_{m-1})$ $t\in I$ is a solution of equation (1) satisfying the initial conditions

$$X\left(t_o,C_o,\dots,C_{m-1}\right)=X_o,\ p^kX\left(t,C_o,\dots,C_{m-1}\right)\Big|_{t=t_0}=X_k,\ k=\overline{1,m-1}. \tag{2}$$

The basic problem consists in constructing the general solution of equation (1).

2. Regular Matrices. The Main Theorem

To construct the general solution of equation (1) we shall need a matrix function of regular matrix.

Definition 3. Matrix $G = G(t),\, t \in I$ with continuous elements will be called a regular matrix if there exists a matrix function $e^{\int G dt}$ definite, continuous and continuously differentiable with respect to *t* on the interval *I*, satisfying the conditions:

$$pe^{\int Gdt} = Ge^{\int Gdt},\ t \in I, \tag{3}$$

$$\exists e^{-\int Gdt}, e^{-\int Gdt} e^{\int Gdt} = e^{\int Gdt} e^{-\int Gdt} = \mathrm{E}, t \in I,$$

where E is a unit matrix.

Remark. Generally speaking, from the regularity of the matrix G the regularity of matrix $-G$ does not follow. Really

$$0 = p\left(e^{\int Gdt} e^{-\int Gdt}\right) = Ge^{\int Gdt} e^{-\int Gdt} + e^{\int Gdt} pe^{-\int Gdt} = G + e^{\int Gdt} pe^{-\int Gdt}.$$

Consequently

$$pe^{-\int Gdt} = -e^{-\int Gdt} G,$$

i.e. if the matrices $e^{-\int Gdt}$ and G are non-permutable, formula (3) is not fulfilled .

Theorem 1 (Basic Theorem). *If the matrices* $A_i = A_i(t), i = \overline{1,m}, t \in I$ *are regular, then the general solution of equation (1) has the form:*

$$X = e^{\int A_m dt}\left(C_0 + \int e^{-\int A_m dt} Y_{m-1} dt\right), t \in I,$$

where

$$Y_k = e^{\int A_k dt}\left(C_{m-k} + \int e^{-\int A_k dt} Y_{k-1} dt\right), k = \overline{1, m-1}, t \in I, \tag{4}$$

$$Y_0 = F, t \in I.$$

Proof. We have

$$(p - A_m)X = (p - A_m)e^{\int A_m dt}\left(C_0 + \int e^{-\int A_m dt} Y_{m-1} dt\right) = Y_{m-1},$$

$$(p - A_{m-1})Y_{m-1} = (p - A_{m-1})e^{\int A_{m-1} dt}\left(C_1 + \int e^{-\int A_{m-1} dt} Y_{m-2} dt\right) = Y_{m-2}, \dots,$$

$$(p - A_1)Y_1 = (p - A_1)e^{\int A_1 dt}\left(C_{m-1} + \int e^{-\int A_1 dt} Y_0 dt\right) = Y_0 = F,\ T \in I.$$

Consequently, the matrix function $X = X(t, C_0, \dots, C_{m-1}), t \in I$ is the solution of equation (1).

Simple calculations give

$$pX = A_m X + Y_{m-1}, t \in I, \Rightarrow Y_{m-1}(t_0) = X_1 - A_m(t_0) X_0,$$

$$p^2 X = (pA_m)X + A_m pX + A_{m-1} Y_{m-1} + Y_{m-2}, t \in I, \Rightarrow$$

$$Y_{m-2}(t_0) = X_2 - (PA_m)\big|_{t=t_0} X_0 - A_m(t_0) X_1 - A_{m-1}(t_0) Y_{m-1}(t_0), \dots,$$

$$Y_0(t_0) = F(t_0),$$

consequently, the constants of matrices $Y_k(t_0), k = \overline{0, m-1}$ are definite.

$$Y_1 = e^{\int A_1 dt}\left(C_{m-1} + \int e^{-\int A_1 dt} F dt\right), t \in I, \Rightarrow C_{m-1} = e^{-\int A_1 dt} Y_1\Big|_{t=t_o} - \int e^{-\int A_1 dt} F dt\Big|_{t=t_o},$$

consequently, the constant of matrix C_{m-1} and the matrix function $Y_1(t), t \in I$ are definite,

$$Y_2 = e^{\int A_2 dt}\left(C_{m-2} + \int e^{-\int A_2 dt} Y_1 dt\right) t \in I, \Rightarrow C_{m-2} = e^{-\int A_2 dt} Y_2\Big|_{t=t_0}, \quad -\int e^{-\int A_2 dt} Y_1 dt\Big|_{t=t_0,}$$

consequently, the constant of matrix C_{m-2} and the matrix function $Y_2(t), t \in I$ are definite,…,

$$Y_{m-1} = e^{\int A_{m-1} dt}\left(C_1 + \int e^{-\int A_{m-1} dt} Y_{m-2} dt\right), t \in I, \Rightarrow C_1 = e^{-\int A_{m-1} dt} Y_{m-1}\Big|_{t=t_0} \quad -\int e^{-\int A_{m-1} dt} Y_{m-2} dt\Big|_{t=t_0},$$

consequently, the constant of matrix C_1 and the matrix function $Y_{m-1}(t), t \in I$ are definite,

$$X = e^{\int A_m dt}\left(C_0 + \int e^{-\int A_m dt} Y_{m-1} dt\right), t \in I, \Rightarrow C_0 = e^{-\int A_m dt} X\Big|_{t=t_0} - \int e^{-\int A_m dt} Y_{m-1}{}^{dt}\Big|_{t=t_0}.$$

Consequently, the constants of matrices $C_k, k = \overline{0, m-1},$ are definite and the solution $X = X(t, C_0, ..., C_{m-1})$ of equation (1) satisfies the initial conditions (2).

Thus Theorem 1 is proved completely. ■

Let us consider the differential equation (1). Let $A_k = a_k = a_k(t), k = \overline{1, m}, t \in I,$ where $a_k(t), k = \overline{i, m},$ are any admissible scalar functions and $F = (f_j^i(t)), i = \overline{1, n}, j = \overline{1, \ell},$ is $n \times \ell$ matrix with continuous elements. Then the differential equation (1) has the form

$$(p - a_1)\left[(p - a_2)\left[...\left[(p - a_{m-1})\left[(p - a_m)X\right]...\right]\ \right]\right] = F. \tag{5}$$

The solution of equation (5) will be called the matrix function $X = X(t) = (x_j^i(t)), i = \overline{1, n}, j = \overline{1, \ell},$ defined on interval I, substitution of which in the equation (5) is admissible, as a result of which we get the identity .

Let t_0 be an arbitrary fixed point of the interval I and $X_{0,} ..., X_{m-1}$ be arbitrary fixed constants of matrix the $n \times \ell$. The matrix function $X(t, C_0, ..., C_{m-1})$ defined on the interval I and depending on arbitrary constants $C_0, ..., C_{m-1}$ of matrix $n \times \ell$ will be called the general solution of equation (5), if $X(t, C_0, ..., C_{m-1}), t \in I$ is a solution of equation (5) satisfying the initial conditions.

$$X(t_{0,} C_0, ..., C_{m-1}) = X_0, \ p^k X(t, C_0, ..., C_{m-1})\Big|_{t=t_0} = X_k, k = \overline{1, m-1}.$$

From Theorem 1 it follows

Theorem 2. *General solution of equation* (5) *has the form:*

$$X = e^{\int a_m dt}(C_0 + \int e^{-\int a_m dt} Y_{m-1} dt),$$

where

$$Y_k = e^{\int a_k dt}(C_{m-k} + \int e^{-\int a_k dt} Y_{k-1} dt), k = \overline{1, m-1},$$

$$Y_0 = F, t \in I. \tag{6}$$

If $\ell = 1,$ we have the differential equation

$$(p - a_1)[(p - a_2)...[(p - a_{m-1})[(p - a_m)\vec{x}]..]] = \overline{F}, \tag{7}$$

where

$$\bar{x} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \vec{F} = \begin{pmatrix} f^1(t) \\ \vdots \\ f^n(t) \end{pmatrix}, t \in I.$$

From Theorem 2 it follows:

Theorem 3. *The general solution of equation (7) has the form*

$$\vec{x} = e^{\int a_m dt}(\vec{C_0} + \int e^{-\int a_m dt}\vec{Y}_{m-1}dt),$$

where

$$\vec{Y}_k = e^{\int a_k dt}(\vec{C}_{m-k} + \int e^{-\int a_k dt}\vec{Y}_{k-1}dt),\ k = \overline{1, m-1},$$

$$\vec{Y}_0 = \vec{F},\ t \in I.$$

Here

$$\vec{C}_k = \begin{pmatrix} c_k^1 \\ \vdots \\ c_k^n \end{pmatrix}, c_k^i = \forall \text{const},\ i = \overline{1,n},\quad k = \overline{0, m-1}.$$

If $n = \ell = 1$, equation (7) has the form

$$(p - a_1)[(p - a_2)...[(p - a_{m-1})[(p - a_m)x]..]] = f\,, \tag{8}$$

where $f = f(t), t \in I$ is a given continuous scalar function.

From Theorem 3 it follows:

Theorem 4. *The general solution of equation (8) has the form:*

$$x = e^{\int a_m dt}(c_0 + \int e^{-\int a_m dt}Y_{m-1}dt),$$

where

$$Y_k = e^{\int a_k dt}(c_{m-k} + \int e^{-\int a_k dt}Y_{k-1}dt),\ k = \overline{1, m-1},$$

$$Y_0 = f,\ t \in I.$$

Here $c_k = \forall$ const, $k = \overline{0, 1-m}.$

3. Applications

If $n = \ell = 1$ and m=2 we have the differential equation

$$\ddot{x} - (a_1 + a_2)\dot{x} + (a_1 a_2 - \dot{a}_2)x = f,\ \dot{} = \frac{d}{dt}. \tag{9}$$

The general solution of equation (9) has the form:

$$x = e^{\int a_2 dt}(c_0 + \int e^{-\int a_2 dt}Ydt),$$

where

$$Y = e^{\int a_1 dt}(c_1 + \int e^{-\int a_1 dt}fdt),\ t \in I. \tag{10}$$

Consider Riccati equation

$$\dot{y} + a(t)y^2 + b(t)y + c(t) = 0,\ t \in I. \tag{11}$$

Let

$$y = \frac{\dot{x}}{a(t)x},\ t \in I. \tag{12}$$

Then we have

$$\ddot{x} - \left(\frac{\dot{a}(t)}{a(t)} - b(t) \right) \dot{x} + a(t)c(t)x = 0, \;\; t \in I. \tag{13}$$

Let

$$\frac{\dot{a}(t)}{a(t)} - b(t) = a_1 + a_2, \; a(t)c(t) = a_1 a_2 - \dot{a}_2, \quad t \in I.$$

Then the general solution of equation (13) has the form (10), where $f \equiv 0$, i.e.

$$x = e^{\int a_2 dt} \left(c_0 + c_1 \int e^{-\int (a_2 - a_1) dt} dt \right), \, t \in I. \tag{14}$$

Consequently the following theorem is valid.

Theorem 5. *The general solution of Riccati equation*

$$\dot{y} + ay^2 + \left(\frac{\dot{a}}{a} - a_1 - a_2 \right) y + \frac{1}{a}(a_1 a_2 - \dot{a}_2) = 0,$$

where $a = a(t), \, a_1 = a_1(t), \, a_2 = a_2(t), \, t \in I$ *are arbitrary admissible functions, has the form* (12), *where x has the form* (14), *i.e.*

$$y = \frac{1}{a} \left(a_2 + \frac{e^{\int (a_1 - a_2) dt}}{c + \int e^{\int (a_1 - a_2) dt} dt} \right), \, c = \frac{c_0}{c_1}, \;\; t \in I.$$

From Theorem 4 it follows:

Theorem 6. *Let us consider the equation*

$$\ddot{x} + a\dot{x} + (\dot{b} - b^2 + ab)x = f, \tag{15}$$

where $a = a(t), \; b = b(t), \; f = f(t), \, t \in I$ *are arbitrary admissible functions. Then the general solution of equation* (15) *has the form:*

$$x = e^{-\int bdt}(c_0 + \int e^{\int bdt} Y dt),$$

where

$$Y = e^{-\int (a-b)dt} \left(c_1 + \int e^{\int (a-b)dt} f dt \right), \, t \in I.$$

From Theorem 6 it follows:

Theorem 7. *The general solution of Hill equation*

$$\ddot{x} + \big(P(t) + \lambda\big) x = 0, \tag{16}$$

where $P(t) + \lambda = \dot{b} - b^2$ *is an arbitrary periodic function, has the form:*

$$x = e^{-\int bdt} \left(c_0 + c_1 \int e^{2\int bdt} dt \right).$$

For example, $P(t) = \cos^2 t + \cos t, \; \lambda = -1$. Then $b = \sin t$ and the general solution of equation

$$\ddot{x} + (\cos^2 t + \cos t - 1)x = 0$$

has the form:

$$x = e^{\cos t} \left(c_0 + c_1 \int e^{-2\cos t} dt \right), \; \forall t.$$

Let us consider the particular case of non-homogeneous hypergeometric equation (see [7])

$$t\ddot{x} + (\beta - t)\dot{x} - \alpha x = g, \tag{17}$$

where α, β are $\forall$ const, $g = g(t)$ is an arbitrary admissible function

If $\beta = 1+\alpha$ and $t \neq 0$ we have

$$\ddot{x} + (1+\alpha - t)t^{-1}\dot{x} - \alpha t^{-1}x = f,\ f = t^{-1}g.$$

From Theorem 6 it follows:

Theorem 8. *If* $\beta = 1+\alpha$ *general solution of equation* (17) *has the form*

$$x = t^{-\alpha}(c_0 + \int t^{\alpha} Y dt),$$

where

$$Y = t^{-1}e^{t}(c_1 + \int e^{-t} g dt),\ \forall t \neq 0.$$

If $g \equiv 0$, we have

$$x = t^{-\alpha}(c_0 + c_1 \int t^{\alpha-1}e^{t} dt),\ \forall t \neq 0.$$

In conclusion let us consider the differential equation (8), where $m = 3$. We have

$$\dddot{x} - (a_1 + a_2 + a_3)\ddot{x} + [a_1(a_2 + a_3) + (a_2 a_3 - \dot{a}_3) - (a_2 + a_3)^{\cdot}]\dot{x} + \\ + [(a_2 a_3 - \dot{a}_3)^{\cdot} - a_1(a_2 a_3 - \dot{a}_3)]x = f. \tag{18}$$

Let us denote $a_2 a_3 - \dot{a}_3 = \alpha,\ a_2 + a_3 = \beta,\ a_1 + a_2 + a_3 = \gamma.$ Then equation (18) has the form

$$\dddot{x} - \gamma\ddot{x} - (\dot{\beta} + \beta^2 - \beta\gamma - \alpha - \gamma)\dot{x} + (\dot{\alpha} + \alpha\beta - \alpha\gamma)x = f. \tag{19}$$

We have $a_1 = \gamma - \beta,\ a_2 = \beta - a_3,\ \dot{a}_3 + a_3^2 - \beta a_3 + \alpha = 0$, i.e. a_3 is an arbitrary solution of Riccati equation

$$\dot{\delta} + \delta^2 - \beta\delta + \alpha = 0. \tag{20}$$

From Theorem 4 it follows:

Theorem 8. *The general solution of equation* (19), *where* $\alpha = \alpha(t),\ \beta = \beta(t), \gamma = \gamma(t), f = f(t), t \in I$ *are arbitrary admissible functions, has the form*

$$x = e^{\int \delta dt}(c_0 + \int e^{-\int \delta dt} Y_2 dt),$$

where δ *is an arbitrary solution of Riccati equation* (20)

$$Y_2 = e^{\int(\beta-\delta)dt}\left(c_1 + \int e^{-\int(\beta-\delta)dt} Y_1 dt\right),$$
$$Y_1 = e^{\int(\gamma-\beta)dt}\left(c_2 + \int e^{-\int(\gamma-\beta)dt} f dt\right), t \in I.$$

If $\gamma = 0, \alpha = \dot{\beta} + \beta^2,\ t \in I,$ equation (19) has the form

$$\dddot{x} + [(\dot{\beta} + \beta^2)^{\cdot} + \beta(\dot{\beta} + \beta^2)]x = f. \tag{21}$$

From Theorem 8 it follows:

Theorem 9. *The general solution of equation* (21), *where* $\beta = \beta(t), t \in I$ *is an arbitrary admissible function, has the form:*

$$x = e^{\int \eta dt}\left(c_0 + \int e^{-\int \eta dt} Y_2 dt\right)$$

where η *is an arbitrary solution of Riccati equation*

$$\dot{\eta} + \eta^2 - \beta\eta + \dot{\beta} + \beta^2 = 0, \tag{22}$$

$$Y_2 = e^{\int(\beta-\eta)dt}\left(c_1 + \int e^{-\int(\beta-\eta)dt} Y_1 dt\right),$$
$$Y_1 = e^{-\int \beta dt}\left(c_2 + \int e^{\int \beta dt} f dt\right), t \in I.$$

If $f \equiv 0,\ t \in I$, we have

$$x = e^{\int \eta dt}\left(c_0 + \int e^{\int(\beta - 2\eta)dt}(c_1 + c_2 \int e^{\int(\eta - 2\beta)dt}dt)dt\right),\ t \in I.$$

From Theorem 4 it follows that if $\beta = a_1 + a_2$ and $\dot\beta + \beta^2 = a_1a_2 - \dot a_2,\ t \in I$,

where $a_1 = a_1(t),\ a_2 = a_2(t),\ t \in I$ are arbitrary admissible functions, then the general solution of Riccati equation (22) has the form

$$\eta = a_2 + \frac{e^{\int(a_1 - a_2)dt}}{c + \int e^{\int(a_1 - a_2)dt}dt},\ t \in I.$$

Consequently from Theorem 9 it follows:

Theorem 10. *If* $\beta = a_1 + a_2$ *and* $\dot\beta + \beta^2 = a_1a_2 - \dot a_2,\ t \in I$, *i.e.* $\dot a_1 + a_1^2 + a_1a_2 + 2\dot a_2 + a_2^2 = 0,\ t \in I$, *where* $a_1 = a_1(t),\ a_2 = a_2(t),\ t \in I$ *are arbitrary admissible functions, then the general solution of the equation*

$$\ddot x + [(a_1a_2 - \dot a_2)^{\cdot} + (a_1 + a_2)(a_1a_2 - \dot a_2)]x = f$$

has the form:

$$x = e_2 s(c_0 + \int e_2^{-1}s^{-1}Y_2 dt),$$

where

$$Y_2 = e_1 s^{-1}(c_1 + \int e_1^{-1}sY_1 dt),$$

$$Y_1 = e_1^{-1}e_2^{-1}(c_2 + \int e_1e_2 f dt),$$

$$e_i = e^{\int a_i dt},\ i = 1, 2,\quad S = \int e_1e_2^{-1}dt,\ t \in I.$$

Remark. We have

$$(a_1a_2 - \dot a_2)^{\cdot} + (a_1 + a_2)(a_1a_2 - \dot a_2) = a_2(\dot a_1 + a_1^2 + a_1a_2 + 2\dot a_2 + a_2^2) -$$
$$- \ddot a_2 - 3a_2\dot a_2 - a_2^3 = -(\ddot a_2 + 3a_2\dot a_2 + a_2^3),\ t \in I.$$

მათემატიკა

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