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Affine Geometry of Hall's *w*-Power Groups

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ABSTRACT. For the Hall's w-power groups coset lattices are constructed. The fundamental theorem of affine geometry is proved. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *affine geometry, w-power group, lattice.*

The notion of a discrete *w*-power group was introduced by F.Hall [1]. At the present time there are a number of works where the properties of *w*-power groups are studied [2]-[10]. In [11] the fundamental theorem of projective geometry for *w*-power groups was proved for principal ideal domains which are not fields.

Let G be only *w*-power group over a binomial ring W. Consider the set $CL(G)$ consisting of all cosets of G with respect to all *w*-subgroups and an empty set $\varnothing$. On $CL(G)$ introduce the partial order:

$$X_1 \subseteq X_2 \Leftrightarrow X_1 \leq X_2$$

for all $X_1, X_2 \in CL(G)$.

Proposition 1. *A set $CL(G)$ is a complete lattice with respect to the operations "$\cup$" and "$\cap$" defined as follows: for $u_\alpha = a_\alpha \cdot A_\alpha$, $\alpha \in J$, $a_\alpha \in G$, $A_\alpha \in CL(G)$ a set-theoretical intersection;*

(i) $\bigcap\limits_{\alpha\in J} u_\alpha$ *is a set-theoretical intersection;*

(ii) $\bigcap\limits_{\alpha\in J} u_\alpha = a_\beta \left\langle A_\alpha, Q_\alpha a_\beta^{-1}, \alpha \in J \right\rangle$,

where β is some fixed index from J.

Let G and G_1 be *w*-power groups over the rings W and W_1 respectively. The bijection $f{:}G\to G_1$ will be called a semilinear isomorphism with respect to the isomorphism $\sigma{:}W\to W_1$ if the equality

$$f(x_1^{\alpha_1} x_2^{\alpha_2}) = f(x_1)^{\sigma(\alpha_1)} f(x_2)^{\sigma(\alpha_2)}$$

is fulfilled for any $x_1, x_2 \in G$ and $\alpha_1, \alpha_2 \in W$ and f will be called a semilinear antiisomorphism if the equality

$$f(x_1^{\alpha_1} x_2^{\alpha_2}) = f(x_2)^{\sigma(\alpha_2)} f(x_1)^{\sigma(\alpha_1)}$$

is fulfilled for any $x_1, x_2 \in G$ and $\alpha_1, \alpha_2 \in W$.

We say that the fundamental theorem of projective geometry is valid for the w-power group G over the ring W if the lattice isomorphism $\varphi: L(G) \to L(G_1)$, where $L(G)$ is the lattice of all w-subgroups of G and G_1 is a w-power group over the ring W_1, implies the existence of a semilinear isomorphism $f{:}G \to G_1$ with respect to the isomorphism $\alpha{:}W \to W_1$ such that $f(A)=\varphi(A)$ for all $A \in L(G)$.

Since in the lattice $CL(G)$ only the elements of G cover $\varnothing$, the isomorphism $f{:}CL(G) \to CL(G_1)$ defines the bijection $f{:}G \to G_1$.

For all the possible isomorphisms f which in the sequel will be called C-isomorphisms we shall select those for which $f(1)=1$. Such isomorphisms will be called natural C-isomorphisms. If $f{:}CL(G) \to CL(G_1)$ is the isomorphism then the bijection φ defined by the equality

$$\varphi(x) = f(x)[f(1)]^{-1}$$

will be a natural C-isomorphism.

We say that the fundamental theorem of affine geometry is valid for w-power group G if any natural C-isomorphism is either a semilinear isomorphism or a semilinear antiisomorphism.

Remark 1. If $a \in A$ is a fixed element and $f(a)=a_1$ then the mapping

$$\varphi(x) = f(ax)[f(a_1)]^{-1}$$

is a natural C-isomorphism. Indeed, φ will be a C-isomorphism defined by the element a_1^{-1} i.e. it will be an automorphism $\left[\tilde{a}_1\right]^{-1} \in Aut[CL(G)]$; since $\varphi(1) = f(a)a_1^{-1}$ we have that f_1 is natural C-isomorphism.

Example. Not each natural C-isomorphism (antiisomorphism) is a semilinear isomorphism. Any onedimentional vector space over $\boldsymbol{Z}_p$ admits $(p$-$1)!$ natural C-automorphisms while the group of internal automorphisms $\boldsymbol{Z}_p$ has order p-1. Therefore for $p>3$ one-dimensional space over $\boldsymbol{Z}_p$ admit natural C-automorphisms different from ordinary ones.

Proposition 2. *Let $CL(G) \to CL(G_1)$ be a natural C-isomorphism. Then the following statements are true.*

(1) *f induces a lattice isomorphism $f{:}\ L(G) \to L(G_1)$;*

(2) $f(\langle M \rangle) = \langle f(M) \rangle$ *for any subset $M \subseteq G$;*

(3) $f(a\langle b \rangle) = f(a)\langle f(b) \rangle$ *for any $a, b \in G$.*

Assume that W is a commutative domain. The W-group G is called torsion-free if $\alpha x=0$ ($\alpha \in W, x \in G$) implies $\alpha=0$ or $x=0$.

Proposition 3. *Let $CL(G) \to CL(G_1)$ be a natural C-isomorphism betweentorsion-free w-power groups over the rings W and W_1. Then*

(a) $f(Z(G))=Z(f(G))$;

(b) *the nilpotency classes on the subgroups coincide;*

(c) *There exists an isomorphism $\sigma{:}W \to W_1$ such that $f(\mu a)=\sigma(\mu)f(a)$, $\mu \in W$, $a \in G$.*

Theorem (Fundamental Theorem of Affine Geometry for W-power Groups). *Let $f{:}CL(G) \to CL(G_1)$ be a natural C-isomorphism between the w-power groups G and G_1 over the rings W and W_1 respectively, then f is either a semilinear isomorphism or a semilinear antiisomorphism with respect to the isomorphism $\sigma: W \to W_1$.*

Remark 2. Thus we have satisfied ourselves that using coset lattices the fundamental theorem of affine geometry can be proved for torsion-free w-power groups over fields, while the fundamental theorem of projective geometry fails [11].

მათემატიკა

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ჰოლის *W*-ხარისხოვანი ჯგუფების აფინური გეომეტრია

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