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Group Representation and Terminal Control of Spatial Rotations

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ABSTRACT. In the article, using spinor representation of orthogonal transformations, the expressions between second order complex unitary transformations matrices and real orthogonal matrices of rotations in L^3 are received, allowing easy calculation of corresponding Eulerian angles. The obtained results have enabled reducing the actually three-dimensional problem of spatial motion control to the one-dimensional problem; control kinematical functions of Eulerian angles and control the spinor matrix of rotation were constructed, by means of which the control process is completely determined.

Keywords: *spinors, control spinor matrix, Hermitian transformations,orthogonal transformations, Eulerian angles, spatial rotations.*

Methods of representation of three-dimensional rotations used in solving various physical and engineering problems are usually confined to the description of individual concrete rotations centered at the origin (zero center) [1]. Among these methods in particular widely (if not uniquely) used is the method of orthogonal real matrices whose elements are functions of Euler angles. However, this method has some disadvantages. The most important of them is, firstly, that the problem of obtaining such a matrix for a concrete rotation is a difficult task in itself and, secondly, that, Euler angles cannot be expressed in terms of the functions of the coordinates of three points – central, initial and terminal – which define the considered rotation [2].

The basic problem arising in this context can be formulated as follows: given two three-dimensional points $x(x_1, x_2, x_3)$ and $y(y_1, y_2, y_3)$, it is required to define the set of all possible transformations and centers of rotations which bring about the transformation of the point x to the point y.

To each vector $x=x_1e_1+x_2e_2+x_3e_3$ of the space L^3 a traceless Hermitian matrix was assigned

$$X = \begin{vmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{vmatrix}, \tag{1}$$

whose elements are the so-called spinor components of the vector x. (1) means that the vector x is identified with Hermitian functionals on the two-dimensional linear space C^2 over the field of complex numbers C. Then for each matrix of form (1) the decomposition exists

$$X=x_1\sigma_1+x_2\sigma_2+x_3\sigma_3, \tag{2}$$

where σ_i are Pauli matrices.

Decomposition (2) permits to introduce $L(C^2)$ - a linear three-dimensional space over the field of real numbers which can be identified with L^3 [3]. The foregoing reasoning implies that for any matrix $C \in C^2$, which is a matrix of transformation between two basis vectors of the space C^2, there also exists a transformation matrix of the corresponding orthonormalized basis vectors in the space L^3 [3].

The problem posed above can be now reformulated in terms of the spinor space C^2: Given two traceless matrices of Hermitian functionals X and Y (they represent initial and final points of rotation) it is required to define the set of unitary matrices $C = \begin{vmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{vmatrix}$ (α and β - complex numbers), which satisfy the equation of spatial rotation

$$Y = \overline{C}^T XC \cdot \tag{3}$$

From equality (3) the following system of linear homogeneous equations with respect to the unknown variables a and b was obtained:

$$\begin{matrix} x_3\alpha + \gamma\beta = y_3\alpha - \overline{\delta\beta} \\ \overline{\gamma}\alpha - x_3\beta = y_3\beta + \overline{\delta}\overline{\alpha} \end{matrix}, \ (\gamma = x_1 + ix_2 \text{ and } \delta = y_1 + iy_2) \tag{4}$$

From the (4) β was defined

$$\operatorname{Re}\beta = \beta_1 = \frac{\alpha_1(x_1 - y_1) + \alpha_2(x_2 + y_2)}{x_3 + y_3} \text{ and } \operatorname{Im}\beta = \beta_2 = \frac{\alpha_2(x_1 + y_1) - \alpha_1(x_2 - y_2)}{x_3 + y_3}. \tag{5}$$

Using the unitarity of the matrix $C(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1)$, we can define either $\alpha_1 = \operatorname{Re}\alpha$ or $\alpha_2 = \operatorname{Im}\alpha$. Note that one of these parameters remains arbitrary, which defines a set of required transformations.

The above permits to define very correspondence relations between the elements of the unitary transformation matrix C in C^2 and the elements of the orthogonal real matrix of rotation A in L^3 was obtained.

$$a_1^1 = (\alpha_1^2 - \alpha_2^2) - (\beta_1^2 - \beta_2^2);\ a_2^1 = 2(\alpha_1\alpha_2 + \beta_1\beta_2);\ a_3^1 = 2(\alpha_2\beta_2 - \alpha_1\beta_1);$$

$$a_1^2 = 2(\beta_1\beta_2 - \alpha_1\alpha_2);\ a_2^2 = (\alpha_1^2 - \alpha_2^2) + (\beta_1^2 - \beta_2^2);\ a_3^2 = 2(\alpha_1\beta_2 + \alpha_2\beta_1);$$

$$a_1^3 = 2(\alpha_1\beta_1 + \alpha_2\beta_2);\ a_3^2 = 2(\alpha_2\beta_1 - \alpha_1\beta_2);\ a_3^3 = (\alpha_1^2 + \alpha_2^2) - (\beta_1^2 + \beta_2^2). \tag{6}$$

Using (6) and the well-known form of orthogonal matrix of rotation A [2], three simple equations for defining Euler angles were obtained:

$$\cos\theta = a_{33};\ \sin\varphi\sin\theta = a_{31} \text{ and } \sin\psi\sin\theta = a_{13}. \tag{7}$$

Thus, the spinor approach to the representation of rotations of a three-dimensional space leads to obtaining simple formulas for calculating elements of real orthogonal matrices and such important parameters as Euler angles as functions of time [4].

Having the foregoing results, it is easy to calculate Euler angles ensuring the rotation of a point $x(x^1, x^2, x^3)$ into a point $y(y^1, y^2, y^3)$. In a general form the control process can be presented as functions of change of Euler angles $\theta(t)$; $\varphi(t)$; $\psi(t)$ that should satisfy the following conditions:

$$\theta(t_0) = 0;\ \phi(t_0) = 0;\ \psi(t_0) = 0,$$

$$\theta(t_f) = \theta_f;\ \phi(t_f) = \phi_f;\ \psi(t_f) = \psi_f, \tag{8}$$

where t_0 and t_f are the initial and final moments of control process.

The fixed vectors $x(x^1,x^2,x^3)$; $y(y^1,y^2,y^3)$ and intermediate rotating vector $\xi(\xi^1,\xi^2,\xi^3)$, which at the initial moment of time $t=t_0$ coincide with an initial vector of rotation $x(x^1,x^2,x^3)$ and at the final one $t=t_f$ - with a final vector $y(y^1,y^2,y^3)$, are introduced. The current angle γ between vectors $x(x^1,x^2,x^3)$ and $\xi(\xi^1,\xi^2,\xi^3)$ at the moment of time $t=t_0$ is equal to zero, and at the moment $t=t_f$– $\gamma = \gamma_f$, where $\gamma_f = ar\cos(\frac{(x,y)}{|x|*|y|}) = ar\cos(\frac{(x,y)}{|x|^2})$; (x,y) – scalar product of vectors x and y.

The coordinates of a rotating intermediate vector $\xi(\xi^1,\xi^2,\xi^3)$ as functions of angle γ were obtained []

$$\xi^i = \frac{|x|^2}{|r|^2}((r\times x)^i \cos(\gamma_f - \gamma(t)) - (r\times y)^i \cos\gamma(t)) \ (i=1,2,3), \tag{9}$$

where $r = x\times y$ - cross product of initial x and final vectors y.

In these expressions an independent variable is the angle γ which can be considered as a function of time, and it means that the coordinates of a vector $\xi(\xi^1,\xi^2,\xi^3)$ also are functions of time. Additionally we assume that $\gamma(t)$ is smooth enough and meets the following conditions: $\gamma(t=t_o)=0$ u $\gamma(t=t_f)=\gamma_f$. We emphasize that the problem of synthesis of spatial movement control is reduced in this way to the definition of a concrete kind of function $\gamma(t)$, that is related to the dynamics of rotation process and that will be stated in future works [4]. In the given work we suppose that $\gamma(t)$ is any function satisfying the above conditions. For certainty let us assume

$$\gamma(t) = \omega t, \text{ where } \omega = 2\pi f \text{ - constant angular frequency.} \tag{10}$$

The vector $\xi(\xi^1,\xi^2,\xi^3)$ is a rotating vector, therefore at each moment of time it can be considered as a final vector of the current moment of rotation process. Having substituted in the formulas (5) instead of coordinates of the point $y(y^1,y^2,y^3)$ expressions (9), we obtain the representations of elements of the spinor matrix C, orthogonal matrix A and Euler angles (7) as functions of time. Thus, we obtain a time-dependent (kinematical) representation of rotation of a point $x(x^1,x^2,x^3)$ into a point $y(y^1,y^2,y^3)$. Before this, however, we should redefine a matrix C in such a manner that at the initial moment of time the spinor equation of rotation (3) would look like $X = \overline{C}^T(t)XC(t)$, that, obviously, is possible only when $C(t)$ is a unitary matrix. It can be done by appropriate selection of parameters α_1 and α_2. Finally we get

$$C(t) = \frac{1}{1+|\beta|^2}\begin{vmatrix} 1 & \frac{((\xi^1(t)-x^1)+i(\xi^2(t)-x^2))}{x^3+\xi^3(t)} \\ \frac{((x^1-\xi^1(t))-i(x^2-\xi^2(t)))}{x^3+\xi^3(t)} & 1 \end{vmatrix}, \tag{11}$$

where $|\beta|^2 = \frac{(x^1-\xi^1(t))^2+(x^2-\xi^2(t))^2}{(x^3+\xi^3(t))^2}$; $\xi^1(t)$, $\xi^2(t)$, $\xi^3(t)$ are the functions of current angle $\gamma(t)$ and therefore also functions of time (they are determined in (11)).

It is obvious that at the initial moment of time t_0 the matrix $C(t=t_o) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, as in this case $\gamma(t_0)=0$ and $\xi(\xi^1,\xi^2,\xi^3) = x(x^1,x^2,x^3)$.

For $t=t_f$ we have $\gamma(t_f)=\gamma_f$, $\xi(\xi^1,\xi^2,\xi^3) = y(y^1,y^2,y^3)$ and, respectively,

$$C(t=t_f)=\frac{1}{1+|\beta|^2}\begin{vmatrix} 1 & \dfrac{((y^1-x^1)+i(y^2-x^2))}{x^3+y^3} \\ \dfrac{((x^1-y^1)-i(x^2-y^2))}{x^3+y^3} & 1 \end{vmatrix}.$$

From the above-said it follows that the spinor matrix of rotation (11) is determined correctly. But in such case the Euler angles (5) are correctly determined also which turn out to be the functions of time

$$\theta(t)=\arccos\left(\frac{(x^3+\xi^3(t))^2-(x^1-\xi^1(t))^2-(x^2-\xi^2(t))^2}{(x^3+\xi^3(t))^2}\right)$$

$$\varphi(t)=\arcsin\left(\frac{2(x^1-\xi^1(t))}{(x^3+\xi^3(t))\sin\theta(t)}\right)$$

$$\psi(t)=\arcsin\left(\frac{2(\xi^1(t)-x^1)}{(x^3+\xi^3(t))\sin\theta(t)}\right). \qquad (12)$$

The obtained expressions (12) solve the problem of determination of the control kinematical functions $\theta(t),\ \phi(t), \psi(t)$. On the other hand, it is necessary to note that the offered theory allows the problem of spatial movement control to be reduced to one-dimensional one. Indeed, it is enough to synthesize in any way a function $\gamma(t)$ meeting the appropriate boundary conditions, then, obviously, the control process will be completely determined by means of the spinor matrix of rotation (11) and functions of Eulerian angles (12).

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REFERENCES

1. *I.M. Gelgamd, R.A. Milnos, Z.A. Shapiro* (1972), Representations of Rotations Group and Lorentz Group, Moscow: Nauka, 425 p.
2. *M. A Shahinpoor* (1988), Robot Engineering Textbook. New York: HARPER & ROW publishers, 527 p.
3. *A.A. Milnikov, A.I. Prangishvili, I.D. Rodonaia* (2005), Spinor Model of Generalized Three-dimensional Rotations. – Automation and Remote Control. Moscow, Vol. 66, No 6, 876-872.
4. *A.A. Milnikov, H. Onal, J. Erguven, I.A. Rodonaia* (2006), Kinematics of Spatial Rotations and Eulerian Angles. – Problems of Mechanics, Tbilisi, 1, 102-108.