საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 2, №1, 2008
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 2, no. 1, 2008

*Mathematics*

# The Modulus of Convexity and Inner Product in Linear Normed Spaces

## George Chelidze

*N. Muskhelishvili Institute of Computational Mathematics, Tbilisi*

(Presented by Academy Member N. Vakhania)

**ABSTRACT. We prove the following result for any real normed space $X$ with $\dim X \geq 3$. Let $a$ be a fixed number from the interval (0,2). The condition $\|x+y\|$ is constant for all $\|x-y\| = a$, $\|x\| = \|y\| = 1$ implies that $X$ is an inner product space. This proposition gives an affirmative answer to Nordlander's conjecture that is related to the modulus of convexity of $X$.** *© 2008 Bull. Georg. Natl. Acad. Sci.*

***Key words***: *inner product spaces, Nordlander's conjecture, modulus of convexity.*

In the present paper we consider a condition under which the norm in a linear normed space $X$ can be expressed by the inner product, i.e. there exists an inner product defined in $X$, such that the following equality holds for all $x$ from $X$: $\|x\|^2 = (x,x)$. There are many properties known for inner product spaces that are not true for all normed spaces. Many of these properties are strong enough restrictions to characterize inner product spaces among normed linear spaces. The main result in this direction is the following Jordan-Neumann characterization ([1]). If for every pair of points $x$ and $y$ the following equality holds

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2),$$

then $X$ is an i.p.s. (inner product space, i.e. Euclidean space). This equality is called parallelogram equality, because, the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides. This fact has its major immediate consequence: a normed linear space $X$ is an i.p.s. iff every two-dimensional linear subspace is Euclidean. Later Day ([2]) proved that for $X$ to be an i.p.s. It is sufficient the parallelogram equality to be held only for norm-one elements of $X$, i.e.

$$\|x+y\|^2 + \|x-y\|^2 = 4$$

for $\forall x, y \in S$, where $S = \{x : x \in X, \|x\| = 1\}$. First generalization of the Jordan-Neumann criterion was given by Aronszajn ([3]). He showed that if there is the real function of three variables $f : R^3 \to R$, such that

$$\|x+y\| = f(\|x-y\|, \|x\|, \|y\|),$$

then $X$ is an i.p.s. Senechalle ([4]) further generalized Aronszajn's and Day's results. He proved that the existence of a real function of one variable $f : R \to R$ guarantees that $X$ is Euclidean space provided the following equality holds

$$\| x+y \| = f(\| x-y \|)$$

for all *x, y* from *S*. In the present paper we prove that if $\dim X \geq 3$, then for characterization of i.p.s. the equality $\| x+y \| = f(\| x-y \|)$ is sufficient only for such vectors $x, y \in S$ which satisfy the condition $\| x-y \| = a$ for some fixed number $a \in (0,2)$. For brevity, let us denote by $C_a$ the set of real numbers: $\{\| x+y \| : x, y \in S, \| x-y \| = a\}$.

**Theorem 1.** *Let X be a real normed space with* $\dim X \geq 3$ *and let a be a fixed number from the interval* (0,2). *The following statements are equivalent:*
(i) $X$ is an inner-product space.
(ii) $C_a$ is a singleton.

This theorem gives an affirmative answer to Nordlander's conjecture that is related to the modulus of convexity of the real normed space *X*. Let us recall this conjecture and the definition of the modulus of convexity of *X*: The modulus of convexity of the space *X* is the function $\delta_X : [0,2] \to [0,1]$ defined by the equality

$$\delta_X(a) = \inf\{1 - \frac{1}{2} \| x+y \| : x, y \in S, \| x-y \| = a\}.$$

Day [5] proved that if

$$\delta_X(a) \geq 1 - \sqrt{1 - \frac{a^2}{4}}, \quad \forall a \in [0,2], \tag{*}$$

then *X* is an i.p.s. Nordlander ([6]) conjectured that it suffices that this inequality holds for some $a \in (0,2)$. Alonso and Benitez ([7]) proved that this assertion is true exactly for $a \in (0,2) \setminus D$, where $D = \{2\cos(k\pi/(2n)) : k = 1, \ldots, n-1; n = 2, 3, \ldots\}$. More precisely, they showed that if $\dim X = 2$, then Nordlander's conjecture is true if $a \in (0,2) \setminus D$ and false if $a \in D$. However, the validity of this conjecture for $\dim X \geq 3$ was open ([8]). Our Theorem 1 gives an affirmative answer to this question. Indeed, it is not hard to show that if (*) is true for some *a*, then the statement (ii) of Theorem 1 is fulfilled as well. This follows from the well-known Day-Nordlander's

**Lemma.** *Let S be the unit sphere of a norm in* $R^2$, $S_a$ *the locus of the midpoints of chords of length* 2*a* (*which is a simple closed curve for* $\forall a \in (0,1)$). *Then the area bounded by* $S_a$ *is* $(1-a^2)$*-times the area bounded by S.*

From (*) it follows that $\frac{\| x+y \|}{2} \leq \sqrt{1 - \frac{a^2}{4}}$ when $\| x-y \| = a$. Using the Lemma we conclude that it cannot be a strong inequality for any pair of points $x, y \in S$, $\| x-y \| = a$, i.e. it should be $\frac{\| x+y \|}{2} = \sqrt{1 - \frac{a^2}{4}}$ and this is the statement (ii) of Theorem 1. So, we have shown that if (*) holds for some $a \in (0,2)$, then the statement (ii) of Theorem 1 holds, i.e. we get

**Theorem 2.** *Let X be a real normed space with* $\dim X \geq 3$. *If* $\delta_X(a) \geq 1 - \sqrt{1 - \frac{a^2}{4}}$ *for some* $a \in (0,2)$, *then X is an inner-product space.*

Supported in part by Grant GNSF/ST06/3-009/

*მათემატიკა*

# ამოზნექილობის მოდული და შიდა ნამრავლი წრფივ ნორმირებულ სივრცეებში

**გ. ჭელიძე**

*ნ. მუსხელიშვილის გამოთვლითი მათემატიკის ინსტიტუტი, თბილისი*

(წარმოდგენილია აკადემიკოს ნ. ვახანიას მიერ)

**მოცემულ ნაშრომში დამტკიცებულია შემდეგი თეორემა: ვთქვათ $X$ წრფივი ნორმირებული სივრცეა $\dim X \geq 3$, ხოლო $a$ - ნებისმიერი ფიქსირებული რიცხვი (0,2) ინტერვალიდან. თუ $\|x+y\|$ არის მუდმივი ყოველი ისეთი $x$ და $y$ ვექტორებისთვის, სადაც $\|x-y\|=a$, $\|x\|=\|y\|=1$, მაშინ $X$ შიდა ნამრავლიანი სივრცეა. ეს დებულება იძლევა დადებით პასუხს ნორდლენდერის ჰიპოთეზაზე, რომელიც შეეხება შიდა ნამრავლიანი სივრცეების დახასიათებას ამოზნექილობის მოდულის მეშვეობით.**

## REFERENCES


1. *P. Jordan & J. von-Neumann* (1935), On inner products in linear metric spaces. – Ann. Math., vol. **36**: 719-723.
2. *M.M. Day* (1947), Some characterizations of inner product spaces. – Trans. Amer. Math. Soc., vol. **62**: 320-337.
3. *N. Aronszajn* (1935), Caracterisation métrique de l'espace de Hilbert, des espaces vectorielles, et de certains groupes métriques. – Comptes rendues Acad. Sci., Paris, vol. **201**: 811-813, 873-875.
4. *D.A. Senechalle* (1968), A characterization of inner product spaces. – Proc. Amer. Math. Soc., vol. **19**: 1306-1312.
5. *M.M. Day* (1944), Uniform convexity in factor and conjugate spaces. – Ann. of Math., (2) **45**: 375-385.
6. *G. Nordlander* (1960), The modulus of convexity in normed linear spaces. – Archiv für Matem., vol. **4**: 15-17.
7. *J.Alonso, C. Benitez* (1988), Some characteristic and non-characteristic properties of inner product spaces, J. Approx. Theory, vol. **55**: 318-325.
8. *D. Amir* (1986), Characterizations of Inner Product Spaces, Basel: Birkhauser Verlag.


*Received September, 2007*